Criteria for elastic fracture

When the non-linear material "process zone" near the crack tip is small compared to all other relevant length scales in the problem / structure, a K-annulus exists and linear elastic fracture mechanis (LEFM) is applicable. This situation is referred to as small scale yielding (SSY).

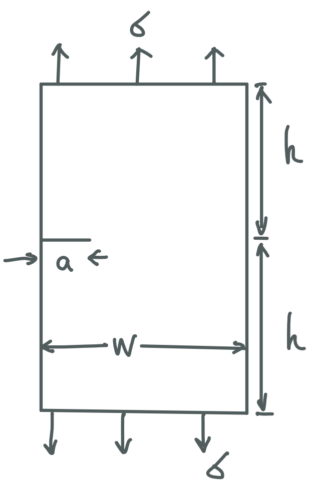
Then what are the conditions required for fracture? Discussions are focused on Mode I here while relevant concepts apply for other modes.

Stress intensity handbooks

Under Mode I conditions, there is an equivalence between K_I and $\mathcal{G}$:

$$\mathcal{G} = \frac{K_I^2}{E'}$$

One can use either $\mathcal{G}$ or K_I to characterize the loading that leads to fracture.



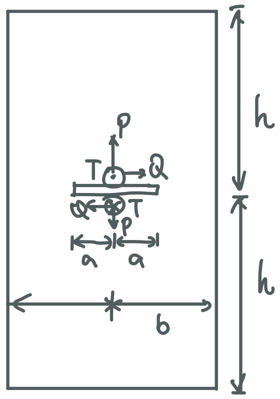
Single edge notch tension

$h/W > 1$

$$K_I = \sigma\sqrt{\pi a}\, F(a/W)$$

$$F(a/W) = 0.265(1-a/W)^4 + \frac{0.857 + 0.265\, a/W}{(1-a/W)^{3/2}}$$

Better than 1% for $\frac{a}{W} < 0.2$, 0.5% for $\frac{a}{W} > 0.2$ (Tada 1973)



$$\begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix} = \frac{1}{\sqrt{\pi a}} \begin{Bmatrix} P \\ Q \\ T \end{Bmatrix} \begin{Bmatrix} F(a/b) \\ F(a/b) \\ F_{III}(a/b) \end{Bmatrix}$$

- $F_{III}(a/b) = \left(\frac{\pi a}{b} \Big/ \sin \frac{\pi a}{b}\right)^{1/2}$
- $F(a/b) = \left[1 - 0.5\left(\frac{a}{b}\right) + 0.957\left(\frac{a}{b}\right)^2 - 0.16\left(\frac{a}{b}\right)^3\right] \Big/ \left(1 - \frac{a}{b}\right)^{1/2}$

Better than 0.3% for any $\frac{a}{b}$ (Asymptotic formula, Tada 1973)

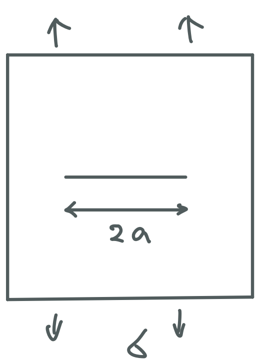
There are also a number of ASTM standard tests such as compact tension (see Fracture Mechanics by A.T. Zehnder, Springer, 2012).

Initiation under monotonic, slow loading

Under ssy situations, fracture occurs when

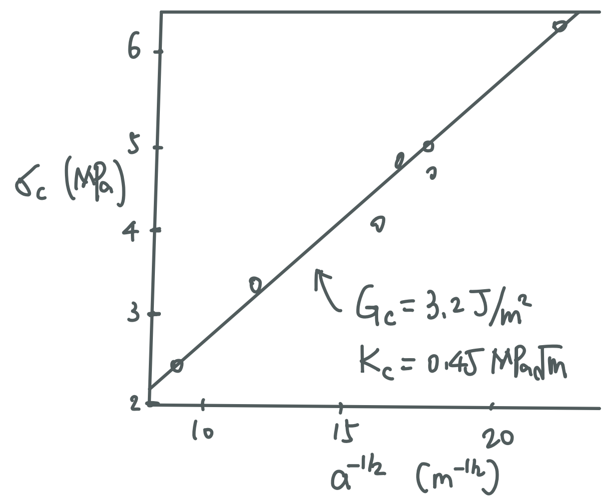
$$\mathcal{G} = \mathcal{G}_c \quad \text{or} \quad K_I = \underbrace{K_c}_{\text{fracture toughness}}.$$

The Griffith's problem is a through crack in a glass plate under tension. Assuming plane stress, $\mathcal{G}$ for this problem is



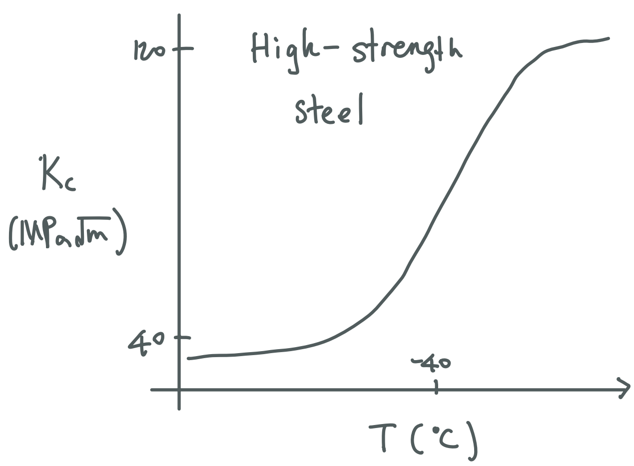
$$K_I = \sigma\sqrt{\pi a} \qquad \mathcal{G} = \frac{\pi\sigma^2 a}{E} \quad \xrightarrow{\text{Initiation}} \quad \sigma_c = \left(\frac{E\mathcal{G}_c}{\pi a}\right)^{1/2} \qquad K_c = (E\mathcal{G}_c)^{1/2}$$

Griffith used glass taken from test tubes blown into thin-walled spheres and cylinders. A glass cutter was used to introduced through cracks in the test sample, which was further annealed to elaminate any residual stresses due to cutting.



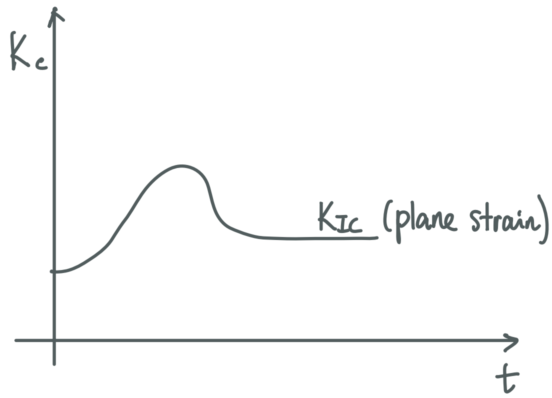
Nominally brittle materials	K_{Ic} ($MPa\sqrt{m}$)	G_c (J/m^2)
Aluminum 7075-T6	25	7800
AlSiC	10	400
Epoxy	0.4	200

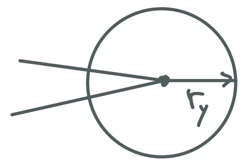
- Temperature effect: For many materials such as metals and polymers, K_c is temperature dependent



The energy needed for fracture drops by a factor of ~10 at low temperatures

- Thickness effect: For small thickness, there is a "loss of constraint" on the plastic zone near the crack tip.





$\sigma_y = \dfrac{K_{Ic}}{\sqrt{2\pi r_y}} \rightarrow r_y \simeq 0.16 \left(\dfrac{K_{Ic}}{\sigma_y}\right)^2$

ASTM condition for a valid K_{Ic} measurement

- $t \gtrsim 2.5 \left(\dfrac{K_{Ic}}{\sigma_y}\right)^2 \simeq 16\, r_y$ ensures plane strain.
- $\mathrm{Min}(a, b, t, \ldots) \gtrsim 2.5 \left(\dfrac{K_{Ic}}{\sigma_y}\right)^2$ ensures ssy.

Al: $K_{Ic} \sim 40\ \mathrm{MPa}\sqrt{\mathrm{m}}$

$\sigma_y \sim 300\ \mathrm{MPa}$

$r_y \sim 2.8\,\mathrm{mm}$

$\rightarrow t \geqslant 44\ \mathrm{mm}$

But for ultrathin sheets, plane-stress description fails. This can be observed directly.

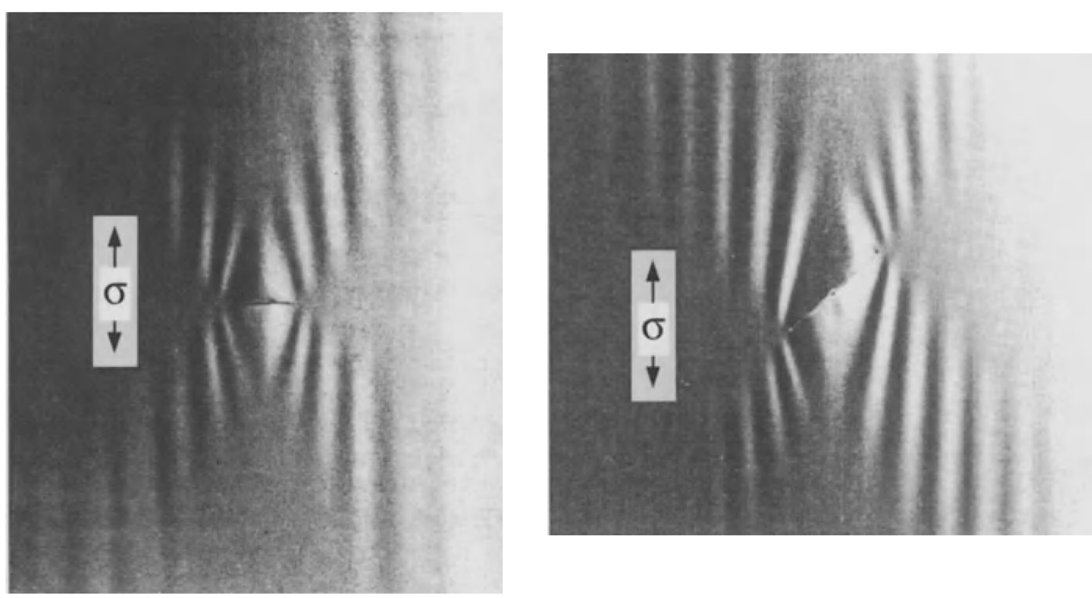
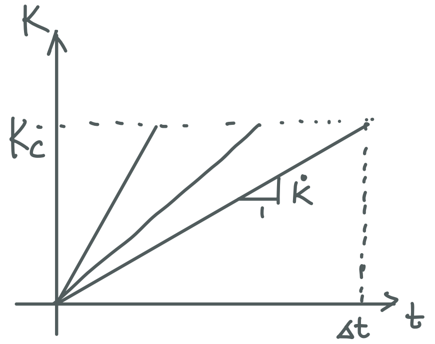


Fig. 3.8. Buckling of a thin plate in uniaxial tension and containing a crack oriented at 90° or at 45°. The orientation of the folds follow closely the stress trajectories, in zones where σ_2 is a compression

- Slow loading : Time to load to failure is much larger than the time for a stress wave to propagate through the material specimen.



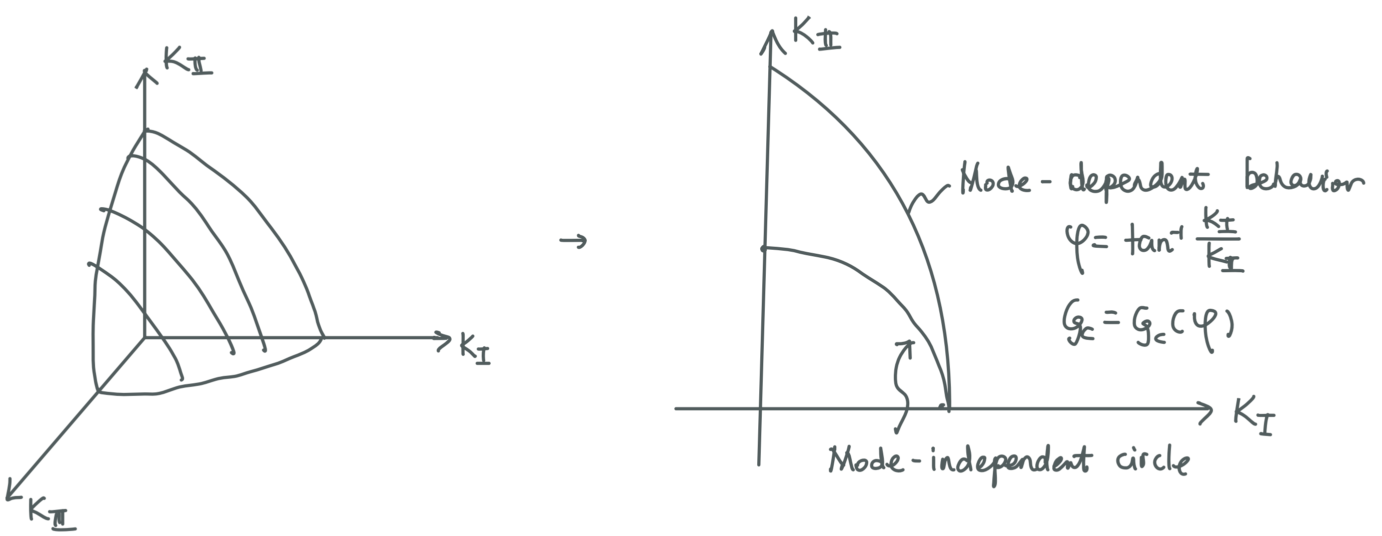
$$\Delta t = \frac{K_c}{\dot{K}} \gg \frac{\mathrm{Max}\,(a, b, t)}{c}, \quad c = \left(\frac{E}{\rho}\right)^{1/2}$$

Speed of sound

$\dot{K}$-dependency is also of importance in viscoelastic, poroelastic materials.

- Mixed mode loading : If $\mathcal{G} = \mathcal{G}_c$ is valid, $\mathcal{G} = \mathcal{G}_c$ represents an ellipsoidal "fracture surface" in K_I, K_{II}, K_{III} space.

$$\mathcal{G} = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} = \mathcal{G}_c$$



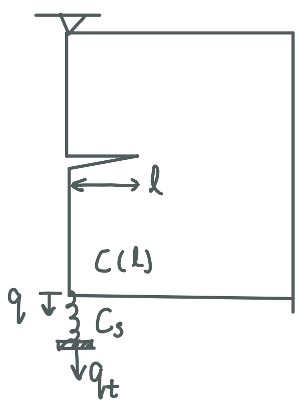
Mode dependent critical fracture energy is especially important in the failure of bimaterial interfaces. Furthermore, crack loaded predominantly in Mode II tend to kink out of the crack plane. The mechanism is attributed to differences in the development of the yield/process zone in different mode (for ductile metals) or frictional effects (absent in mode I but present in mode II & III for brittle interfaces).

The "simple" criteria above let us determine whether a crack will grow or not, but they do not tell us anything about how fast, how far, or in what direction the crack will grow.

Crack growth stability (how far?)

We have shown in the analysis of the DCB speciment that $\mathcal{G} \propto a^2$ for fixed load and $\mathcal{G} \sim a^{-4}$ for fixed displacement. The crack growth is likely stable in the latter case since $\frac{d\mathcal{G}}{da} < 0$. There are additional stablizing mechanisms.

- Loading by compliant systems



In many applications, the material/structure is not under fixed displacement or fixed force loadings, but an intermediate state — a fixed q_t (total displacement)

q: displacement of the body of stiffness $C(\ell)$

q_s: displacement of the spring of stiffness C_s

p: the force on the body / elastic spring

$$p = Cq = C_s q_s \,, \quad q_t = q + q_s \;\rightarrow\; q_t = \left(\frac{1}{C} + \frac{1}{C_s}\right) p$$

Note that $C_s \to \infty \Leftrightarrow$ fixed displacement

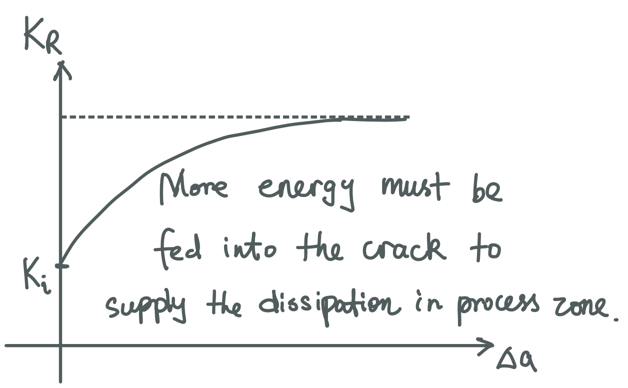
$$U\Big|_{q_t} = \frac{1}{2}Cq^2 + \frac{1}{2}C_s q_s^2 = \frac{1}{2}\frac{p^2}{C} + \frac{1}{2}\frac{p^2}{C_s} = \frac{1}{2}\left(\frac{1}{C}+\frac{1}{C_s}\right) q_t^2 \Big/ \left(\frac{1}{C}+\frac{1}{C_s}\right)^2 = \frac{1}{2} q_t^2 \Big/ \left(\frac{1}{C}+\frac{1}{C_s}\right)$$

$$G = -\frac{\partial U}{\partial l}\Big|_{q_t} = \frac{1}{2} q_t^2 \underbrace{\frac{-\frac{1}{C^2}\frac{\partial C}{\partial l}}{\left(\frac{1}{C}+\frac{1}{C_s}\right)^2}}_{\frac{C_s^2}{(C+C_s)^2}\left(-\frac{\partial C}{\partial l}\right)} = -\frac{1}{2}\frac{p^2}{C^2}\frac{\partial C}{\partial l} = -\frac{1}{2}q^2\frac{\partial C}{\partial l} \quad \text{(identical to the case of fixed } q\text{)}$$

You may find this system is also eventually stable as l increases ($p\downarrow$).

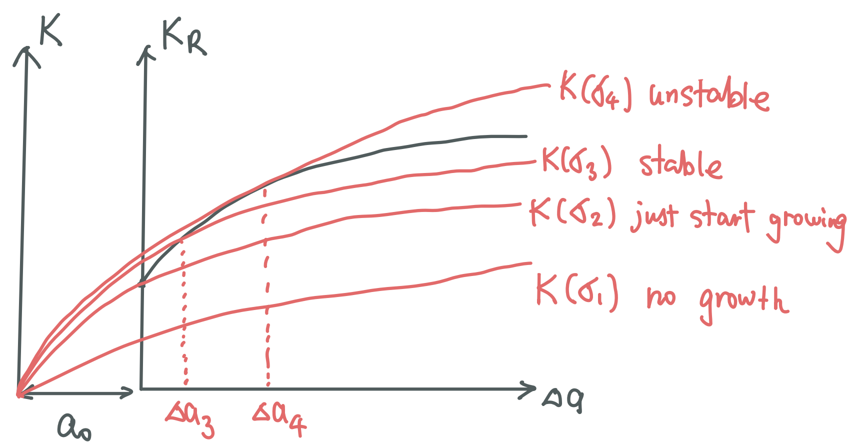
- Resistance curve (R– curve)

In some cases, crack growth can be stable even when $\frac{\partial G}{\partial l} > 0$. How so? For brittle materials, G_c is a constant. For ductile materials (particular thin sheets), it is generally found that G_c increases as the crack grows.



$\Delta a = a - a_0$

Current crack length — Initial crack length

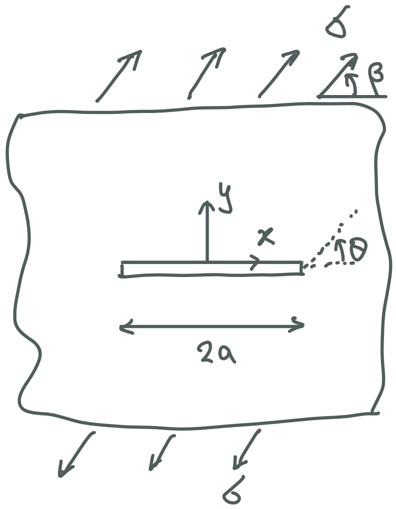


$K \sim \sigma\sqrt{\pi a}\, f(a/w)$

- Crack occurs when $K_{App}(\sigma_1, a_0) \geq K_R(\Delta a = 0)$

- Crack is unstable when $\frac{dK_{App}}{da}\Big|_a > \frac{dK_R}{d(\Delta a)}\Big|_{\Delta a = a - a_0}$

Mixed-mode fracture initiation & growth (Which direction?)



Subject to a combination of Mode-I and Mode-II loadings, crack will generally not propagate straight ahead (unless $\theta = 0$ is a weak interface).

Our previous analysis has been on $\mathcal{G} = -\left.\frac{\partial W}{\partial a}\right|_{\theta=0}$. What if $\theta \neq 0$? There are a few theories such as ① maximum hoop stress ② maximum energy release rate.

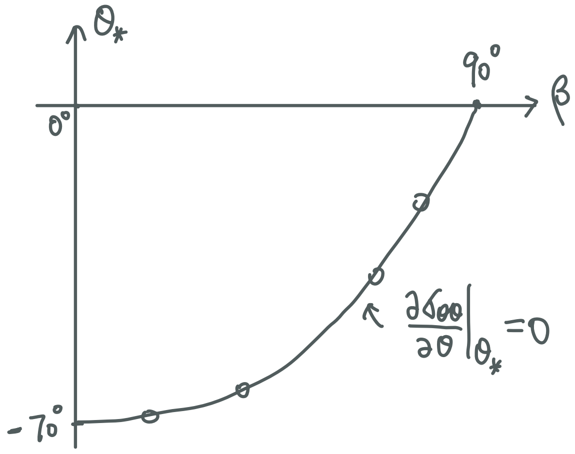
① Maximum hoop stress theory: Crack will grow in the direction θ_* of maximum hoop stress, when $\sqrt{r}\,\sigma_{\theta\theta}(r, \theta_*) \geq$ Constant.

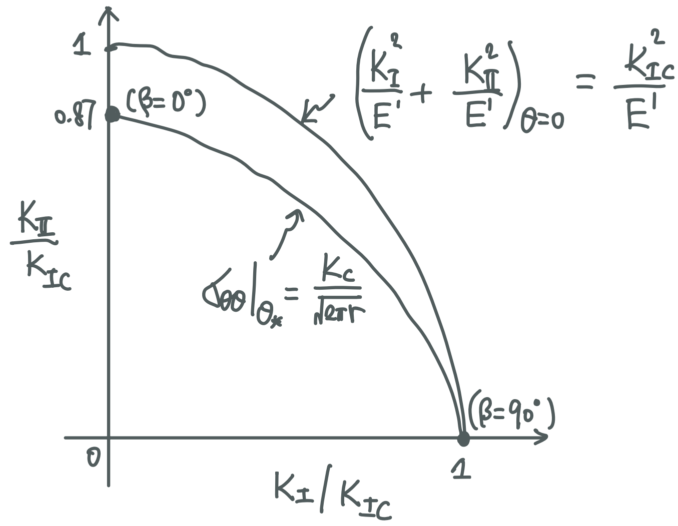
$$\rightarrow \left.\frac{\partial \sigma_{\theta\theta}}{\partial \theta}\right|_{\theta_*} = 0, \quad \left.\frac{\partial^2 \sigma_{\theta\theta}}{\partial \theta^2}\right|_{\theta_*} < 0, \quad \sigma_{\theta\theta}(r, \theta_*) \geq \frac{K_{IC}}{\sqrt{2\pi r}}$$

For this problem $K_I = \sigma\sqrt{\pi a}\,\sin\beta$, $K_{II} = \sigma\sqrt{\pi a}\,\cos\beta$

$$\sigma_{\theta\theta}(r,\theta) = \frac{1}{\sqrt{2\pi r}}\left(K_I \cos\frac{\theta}{2}\,\frac{1+\cos\theta}{2} - K_{II}\,\frac{3}{2}\sin\theta\cos\frac{\theta}{2}\right) + O(r^0)$$

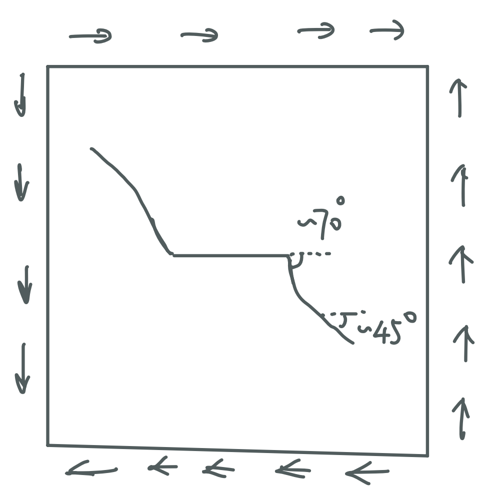
- Under pure Mode II ($\beta = 0^\circ$), the crack will grow at $\theta_* = -70.6^\circ$ at a stress intensity factor of $K_{II} = 0.87\,K_{IC}$. [Allowing kinking leads to a slightly smaller K_{II}]



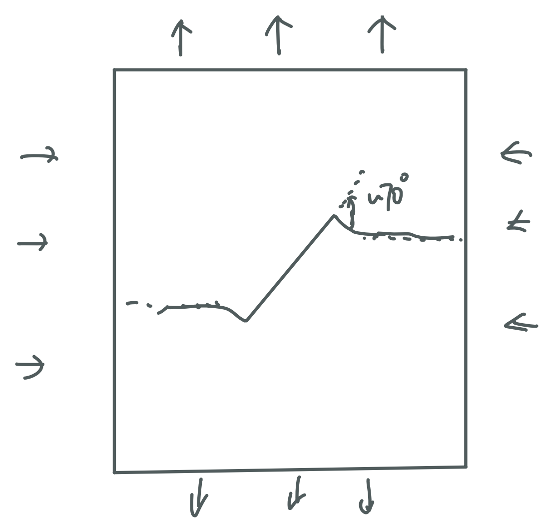


② Maximum energy release rate criterion: $\left.\frac{\partial \mathcal{G}}{\partial \theta}\right|_{\theta_*} = 0$, $\left.\frac{\partial^2 \mathcal{G}}{\partial \theta^2}\right|_{\theta_*} < 0$, $\mathcal{G}(\theta_*) = \mathcal{G}_c$

$\rightarrow$ $\theta_* = -75.6^\circ$ at $K_{II} = 0.817\, K_{IC}$ under pure Mode II loading.



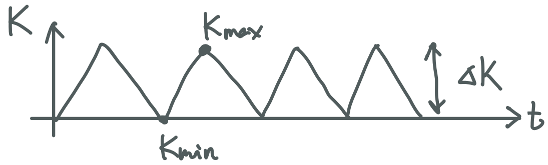
or



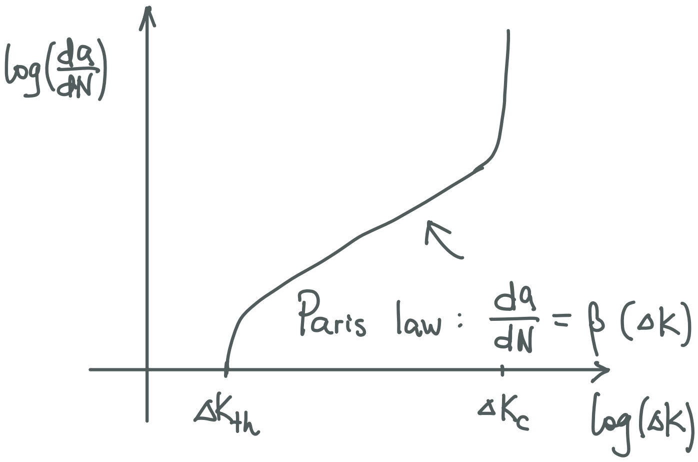
Why -70° not -45°? The presence of crack disturbs the stress field. The crack path will evolve to lie along the plane of maximum principle stress.

Mode I cyclic loading (Fatigue)

Under cyclic loading, materials can fail due to fatigue at stress levels well below their static strength.



$a = a(N)$



Paris law: $\frac{da}{dN} = \beta (\Delta K)^n$, $n \sim 2-4$

- $\Delta K < \Delta K_{th}$ (threshold), no crack growth
- $\Delta K > \Delta K_c$, Immediate failure
- $\Delta K_{th} < \Delta K < \Delta K_c$, crack length a increases over N

  failure occurs when $K_{max}(a) = K_{IC}$

The authors are hesitant but cannot resist the temptation to draw the straight line of slope $^1/_4$ through the data in Fig. 4. The equation of this line is observed to be

$$\frac{da}{dN} = \frac{(\Delta k)^4}{M} \tag{35}$$

Equation (35) fits the data almost as well as McEvily and Illg's extended law, equation (14), and is considerably simpler in form.

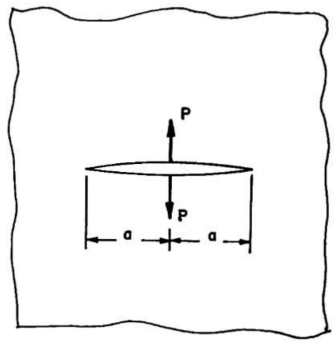


Fig. 5 Wedge-force test configuration

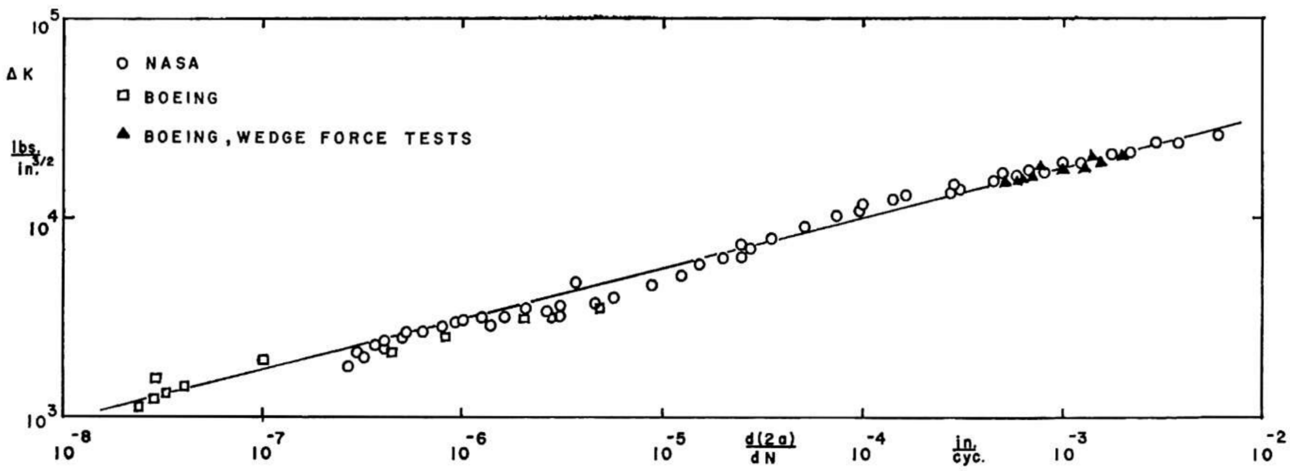
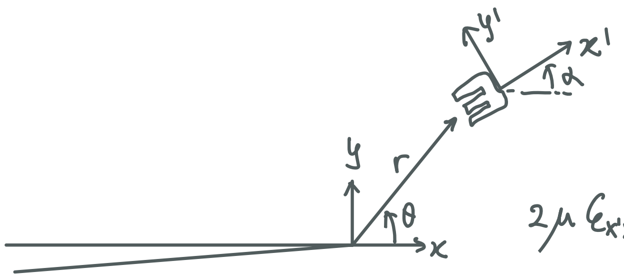


Fig. 6 Broad trend of crack-growth data on 7075-T6 aluminum alloy including wedge-force tests

Experimental methods

In some cases, it may not be pratical to determine stress intensity factors from analytical or computational methods. For example, perhaps the loading is unknown, or is dynamic. One may wish to determine the SIF experimentally, based on local measurement of stress, strain and displacement. There are a number of expermental methods that have been developed, including photoelasticity, interferometry, digital image correlation, strain guages and so on (see Chapter 5 in Fracture Mechanics by A.T. Zehnder)

Only strain guages method by Dally and Stanford (1987) is discussed here.



HOTs may be important:

$$2\mu \varepsilon_{x'x'} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1-\nu}{1+\nu}\cos\frac{\theta}{2} - \frac{1}{2}\sin\theta \sin\frac{3\theta}{2}\cos 2\alpha + \frac{1}{2}\sin\theta\cos\frac{3\theta}{2}\sin 2\alpha\right]$$

$$+ A_0\left[\frac{1-\nu}{1+\nu} + \cos 2\alpha\right]$$

$$+ A_{1/2} r^{1/2}\cos\frac{\theta}{2}\left[\frac{1-\nu}{1+\nu} - \sin^2\frac{\theta}{2}\cos 2\alpha + \frac{1}{2}\sin\theta\sin 2\alpha\right] + O(r)$$

K_I, A_0, $A_{1/2}$ — Three parameters, but only one measurement. What do you do?