Westergaard's stress function

Recall method of complex variables

$z = x + iy = re^{i\theta}$, r – modulus of z . θ - argument

$\bar{z} = x - iy$ complex conjugate

$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$, $\overline{f(z)} = \bar{f}(\bar{z}) = u(x,y) - i\,v(x,y)$

f is differentiable at point z_0 in D if

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

exists and is independent of HOW $\Delta z \to 0$. If f is differentiable at all points in a domain D, then it is said to be analytic or regular in D.

$$\begin{cases} z = x + iy \\ \bar{z} = x - iy \end{cases} \rightarrow \begin{cases} \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z}\cdot\dfrac{\partial z}{\partial x} + \dfrac{\partial}{\partial \bar{z}}\dfrac{\partial \bar{z}}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}} \\ \dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial z}\dfrac{\partial z}{\partial y} + \dfrac{\partial}{\partial \bar{z}}\dfrac{\partial \bar{z}}{\partial y} = i\dfrac{\partial}{\partial z} - i\dfrac{\partial}{\partial \bar{z}} \end{cases}$$

$$\Rightarrow$$

$$\begin{cases} x = \frac{1}{2}(z+\bar{z}) \\ y = \frac{1}{2i}(z-\bar{z}) \end{cases} \rightarrow \begin{cases} \dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right) \\ \dfrac{\partial}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right) \end{cases}$$

$$\rightarrow \boxed{\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \bar{z}^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} , \quad \frac{\partial^2}{\partial y^2} = -\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \bar{z}^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} , \quad \frac{\partial^2}{\partial x \partial y} = i\frac{\partial^2}{\partial z^2} - i\frac{\partial^2}{\partial \bar{z}^2}}$$

$$f'(z) = \frac{\partial}{\partial z}(u+iv) = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2i}\frac{\partial u}{\partial y} + i\left(\frac{1}{2}\frac{\partial v}{\partial x} + \frac{1}{2i}\frac{\partial v}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$f(z)$ analytic requires $f'(z)$ the same regardless of the path of Δz, e.g. Δx or Δy

$$\rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

First, consider the Mode III (anti-plane shear) problem, which can be formulated as

$$\text{Equilibrium equations:}\quad \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0$$

$$\text{Kinematics:}\quad \gamma_{xz} = \frac{\partial W}{\partial x}, \quad \gamma_{yz} = \frac{\partial W}{\partial y}$$

$$\text{Material law:}\quad \sigma_{xz} = \mu \gamma_{xz}, \quad \sigma_{yz} = \mu \gamma_{yz}$$

It is convenient to introduce stress function φ, such that

$$\sigma_{xz} = -\frac{\partial \psi}{\partial y}, \quad \sigma_{yz} = \frac{\partial \psi}{\partial x} \quad \text{(equilibrium satisfied automatically).}$$

φ is not abitrary since $\dfrac{\partial \gamma_{xz}}{\partial y} \equiv \dfrac{\partial \gamma_{yz}}{\partial x}$, i.e.,

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0} \quad \text{Harmonic equation}$$

Harmonic equation $\rightarrow$ $4\dfrac{\partial^2 \psi}{\partial z \partial \bar{z}} = 0$.

has to equal $\bar{w}_1(\bar{z})$ to ensure φ is real

Immediately, the solution is
$$\begin{aligned}
\psi &= \frac{1}{2}\left[\omega_1(z) + w_2(\bar{z})\right] \\
&= \frac{1}{2}\left[w(z) + \overline{w(z)}\right] \\
&= \mathrm{Re}\left[\omega(z)\right]
\end{aligned}$$

$$\sigma_{xz} = -\frac{\partial \psi}{\partial y} = -i\frac{\partial \psi}{\partial z} + i\frac{\partial \psi}{\partial \bar{z}} = -\frac{i}{2}\omega'(z) + \frac{i}{2}\frac{\partial \overline{w(z)}}{\partial \bar{z}} \equiv -\frac{i}{2}w'(z) + \frac{i}{2}\overline{w'(z)}$$

$$\sigma_{yz} = \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial z} + \frac{\partial \psi}{\partial \bar{z}} = \frac{1}{2}\omega'(z) + \frac{1}{2}\overline{w'(z)}$$

$\rightarrow \sigma_{yz} + i\sigma_{xz} = \omega'(z)$

σ_{yz}: $\mu \frac{\partial W}{\partial y}$, σ_{xz}: $\mu \frac{\partial W}{\partial x}$

ω is analytic, complex stress function

$$\rightarrow \mu\left(i\frac{\partial W}{\partial z} - i\frac{\partial W}{\partial \bar{z}}\right) + i\mu\left(\frac{\partial W}{\partial z} + \frac{\partial W}{\partial \bar{z}}\right) = 2i\mu\frac{\partial W}{\partial z} = \omega'(z) \rightarrow W = \frac{-i}{2\mu}\omega(z) + \underbrace{f(\bar{z})}_{\frac{i}{2\mu}\overline{\omega(z)}}$$

$$\rightarrow W = \frac{1}{\mu}\,\mathrm{Im}\left[\omega(z)\right]$$

Westergaard's stress function (Mode III) : $Z_{III} = \omega'(z)$, $\hat{Z}_{III} = \omega(z)$

$$\sigma_{yz} = \mathrm{Re}\left[Z_{III}(z)\right]$$

$$\sigma_{xz} = \mathrm{Im}\left[Z_{III}(z)\right]$$

$$W = \frac{1}{\mu}\,\mathrm{Im}\left[\hat{Z}_{III}(z)\right]$$

Will show properties of Z_{III} later. Let's finalize Z_I & Z_{II} first.

For plane stress/strain problems, the governing equation is biharmonic:

$$\nabla^2\nabla^2\Phi = 0 \longrightarrow 16\frac{\partial^4\Phi}{\partial z^2\partial\bar{z}^2} = 0$$

Similarly, $\frac{\partial^2\Phi}{\partial z\partial\bar{z}} = f_1(z) + g_1(\bar{z})$, $\frac{\partial\Phi}{\partial z} = \bar{z}f_1(z) + \underbrace{g(\bar{z})}_{g' = g_1} + h_1(z)$

$$\Phi = \bar{z}\overbrace{f(z)}^{f' = f_1} + z\,g(\bar{z}) + \overbrace{h(z)}^{h' = h_1} + k(\bar{z})$$

$$= \bar{z}f(z) + \overline{\bar{z}f(z)} + h(z) + \overline{h(z)} \quad \leftarrow \Phi \text{ is Real.}$$

$$= \mathrm{Re}[\bar{z}\underbrace{\phi(z)}_{2f(z)} + \underbrace{G(z)}_{2h(z)}]$$

$$\sigma_{xx} + \sigma_{yy} = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 4\frac{\partial^2\Phi}{\partial z\partial\bar{z}} = 4\,\mathrm{Re}[\phi'(z)]$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = \frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial y^2} - 2i\frac{\partial^2\Phi}{\partial x\partial y} = 2\frac{\partial^2\Phi}{\partial z^2} + 2\frac{\partial^2\Phi}{\partial\bar{z}^2} - 2i\left(i\frac{\partial^2\Phi}{\partial z^2} - i\frac{\partial^2\Phi}{\partial\bar{z}^2}\right)$$

$$= 4\frac{\partial^2\Phi}{\partial z^2}$$

$$= 4\bar{z}f''(z) + 4\overbrace{h''(z)}^{\frac{1}{2}\psi'(z)}$$

$$= 2[\bar{z}\phi''(z) + \psi'(z)]$$

where $\phi(z)$, $\psi(z)$ are complex potentials.

You should be able to show: $2\mu(u+iv) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}$

$$\rightarrow \quad \sigma_{xx} = \mathrm{Re}\left[2\phi'(z) - \bar{z}\phi''(z) - \psi'(z)\right]$$

$$\sigma_{yy} = \mathrm{Re}\left[2\phi'(z) + \bar{z}\phi''(z) + \psi'(z)\right]$$

$$\sigma_{xy} = \mathrm{Im}\left[\bar{z}\phi''(z) + \psi'(z)\right]$$

Consider cracks on the x-axis with symmetric loading such that.

$$\sigma_{xx}(x,y) = \sigma_{xx}(x,-y), \quad \sigma_{yy}(x,y) = \sigma_{yy}(x,-y), \quad \underbrace{\sigma_{xy}(x,y) = -\sigma_{xy}(x,y)}$$

$$\rightarrow \sigma_{xy}(x, y=0) = 0$$

Reorganize:

$$\sigma_{xx} = \mathrm{Re}\left[2\phi'(z) + \overbrace{(z-\bar{z})}^{2iy}\phi''(z) - z\phi''(z) - \psi'(z)\right]$$

$$\sigma_{yy} = \mathrm{Re}\left[2\phi'(z) - (z-\bar{z})\phi''(z) + z\phi''(z) + \psi'(z)\right]$$

$$\sigma_{xy} = \mathrm{Im}\left[-(z-\bar{z})\phi''(z) + z\phi''(z) + \psi'(z)\right]$$

Since $z\phi''(z)$ is an analytic function, we can always take $\psi'(z) = -z\phi''(z)$, and we assure that equations of elasticity are satisfied. However, the solutions these functions generate only satisfy a limited set of boundary conditions. In particular, they have

$$\sigma_{xy}(x, y=0) = 0 \quad \& \quad \sigma_{xx}(x, y=0) = \sigma_{yy}(x, y=0)$$

Define Mode I Westergaard stress function as $Z_I(z) = 2\phi'(z)$

$$\sigma_{xx} = Re\left[Z_I(z) + iy Z_I'(z)\right] = Re\left[Z_I(z)\right] - y\, Im\left[Z_I'(z)\right]$$

$$\sigma_{yy} = Re\left[Z_I(z) - iy Z_I'(z)\right] = Re\left[Z_I(z)\right] + y\, Im\left[Z_I'(z)\right]$$

$$\sigma_{xy} = Im\left[-iy Z_I'(z)\right] = -y\, Re\left[Z_I'(z)\right]$$

Useful for Mode I solutions for cracks on the x-axis in infinite 2D spaces.

Next, consider mode II type loadings which we showed dictates anti symmetry:

$$\underbrace{\sigma_{xx}(x,y) = -\sigma_{xx}(x,-y), \quad \sigma_{yy}(x,y) = -\sigma_{yy}(x,-y)}, \quad \sigma_{xy}(x,y) = \sigma_{xy}(x,-y)$$

On intact regions: $\sigma_{xx}(x,y=0) = \sigma_{yy}(x,y=0) = 0$

On traction-free crack faces : $\sigma_{xx}(x,y=0) \neq 0$, $\sigma_{yy}(x,y=0) = 0$

$$\sigma_{yy} = Re\left[2\phi'(z) - \overbrace{(z-\bar{z})}^{2yi}\phi''(z) + z\phi''(z) + \psi'(z)\right]$$

→ Take $2\phi'(z) = -z\phi''(z) - \psi'(z)$. Again, $\psi(z)$ is analytic, i.e., equations of elasticity are satisfied.

Define the mode II Westergaard stress function as $Z_{II} = i2\phi'(z)$

$$\sigma_{yy} = -y\, Re\left[Z_{II}'(z)\right]$$

$$\sigma_{xx} = Re\left[-2i Z_{II}(z) + y Z_{II}'(z)\right] = 2\, Im\left[Z_{II}(z)\right] + y\, Re\left[Z_{II}'(z)\right]$$

$$\sigma_{xy} = Im\left[-y Z_{II}'(z) + i Z_{II}(z)\right] = Re\left[Z_{II}(z)\right] - y\, Im\left[Z_{II}'(z)\right]$$

Useful for cracks on the x-axis in infinite 2D space with Mode II type loading

Displacement fields

$$2\mu\, u_x = \tfrac{1}{2}(\kappa-1)\,\mathrm{Re}\big[\hat{Z}_{I}(z)\big] - y\,\mathrm{Im}\big[Z_{I}(z)\big] + \tfrac{1}{2}(\kappa+1)\,\mathrm{Im}\big[\hat{Z}_{II}(z)\big] + y\,\mathrm{Re}\big[Z_{II}(z)\big]$$

$$2\mu\, u_y = \tfrac{1}{2}(\kappa+1)\,\mathrm{Im}\big[\hat{Z}_{I}(z)\big] - y\,\mathrm{Re}\big[Z_{I}(z)\big] - \tfrac{1}{2}(\kappa-1)\,\mathrm{Re}\big[\hat{Z}_{II}(z)\big] - y\,\mathrm{Im}\big[Z_{II}(z)\big]$$

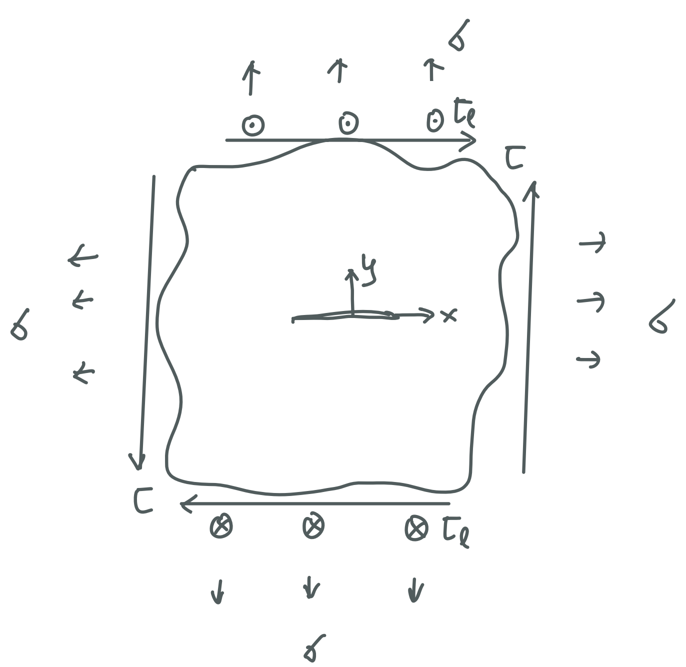
where $Z_I(z) = \dfrac{d\hat{Z}_I}{dz}$, $Z_{II}(z) = \dfrac{d\hat{Z}_{II}}{dz}$, and again $\kappa = \begin{cases} 3-4\nu, & \text{plane strain} \\ \frac{3-\nu}{1+\nu}, & \text{plane stress.} \end{cases}$

With these definition of Westergaard stress functions, it turns out many problems have similar solutions in different modes. For example, the asymptotic solutions look like:

$$\begin{Bmatrix} Z_I(z) \\ Z_{II}(z) \\ Z_{III}(z) \end{Bmatrix} = \begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix} \frac{1}{\sqrt{2\pi z}}$$

(Z_I, Z_{II}, Z_{III} have dimension of stress)

Example:



K_I, K_{II}, K_{III}?

Let's look at mode I first.

Boundary conditions (Note $\sigma_{yy} = \text{Re}\, Z_I + y\, \text{Im}\, Z_I'$, $\sigma_{xx} = \text{Re}\, Z_I - y\, \text{Im}\, Z_I'$)

- $\sigma_{xy}(|x|<a,\ y=0) = 0$ ✓ Satisfied automatically by Z_I

- $\sigma_{yy}(|x|<a,\ y=0) = 0 \rightarrow \text{Re}\, Z_I \Big|_{|x|<a,\ y=0} = 0$

How to ensure Z_I imaginary for $|x|<a$? $\rightarrow \sqrt{x^2-a^2}$ or $\sqrt{z^2-a^2}$

- As we approach the crack, i.e., $|z| \rightarrow a^+$, we expect $r^{-1/2}$ singularities.

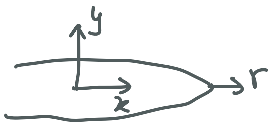
$$\rightarrow Z_I \propto \frac{1}{\sqrt{z^2-a^2}}$$

- $\sigma_{xx} = \sigma_{yy} = \sigma$ as $r = |z| \rightarrow \infty$. This requires $Z_I \sim \dfrac{z \cdot \sigma}{\sqrt{z^2-a^2}}$. Indeed,

$$\boxed{Z_I = \frac{\sigma z}{\sqrt{z^2-a^2}}}$$

We would also find: $Z_{II} = \dfrac{\tau z}{\sqrt{z^2-a^2}}$, $Z_{III} = \dfrac{\tau_l z}{\sqrt{z^2-a^2}}$

Determine K_I: $\sigma_{yy}(x>a,\ y=0) = \text{Re}\, Z_I \Big|_{x>a,\ y=0} = \dfrac{\sigma x}{\sqrt{x^2-a^2}}$



$x = r + a$ $\quad K_I = \lim_{r\to 0} \sqrt{2\pi r}\, \sigma_{yy}(r) = \lim_{r\to 0} \sqrt{2\pi r} \cdot \dfrac{\sigma(r+a)}{\sqrt{(r+2a)\cdot r}} = \sigma\sqrt{\pi a}$ ✓

Similarly we would find $K_{II} = \tau\sqrt{\pi a}$, $K_{III} = \tau_l \sqrt{\pi a}$

Branch cut (分支切割)

We have focused on $x<a$, but when dealing with $x<a$, Z_I is double-(multi-) valued. Need to use branch cut(s) through branch points. We often have the following two scenarios:

Semi-infinite cracks

$$\sqrt{z} = \sqrt{r}\, e^{i\theta/2}$$

Let's compute $z = 1+i$

$$r \equiv \sqrt{2},\quad \theta = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17}{4}\pi \dots$$

$$\to \sqrt{z} = 2^{\frac{1}{4}}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right),$$

$$-2^{\frac{1}{4}}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right),$$

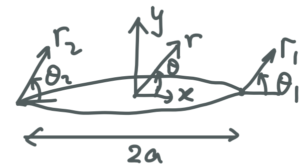
$$+2^{\frac{1}{4}}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right),$$

. . . .

May naturally take the negative x axis as the branch cut

$\to -\pi \le \theta \le \pi \to \sqrt{z}$ single-valued

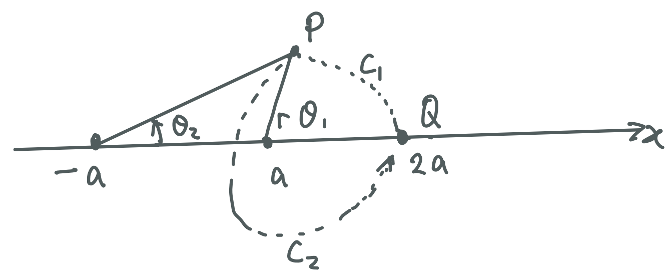
Center cracks



$$z-a = r_1 e^{i\theta_1},\quad z+a = r_2 e^{i\theta_2}$$

$$\to \sqrt{z^2-a^2} = \sqrt{r_1 r_2}\, e^{i\frac{\theta_1+\theta_2}{2}}$$

- No branch cuts

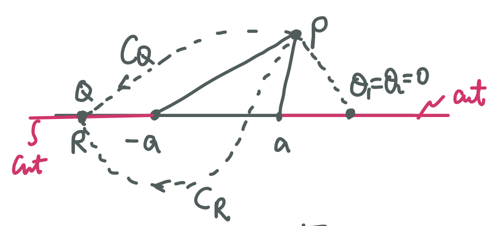


Suppose path C_1 gives $\theta_1 = \theta_2 = 0$

Then path C_2 gives $\theta_2 = 0$, $\theta_1 = 2\pi$

$$\to \sqrt{z^2-a^2}\Big|_{C_1} = -\sqrt{z^2-a^2}\Big|_{C_2}$$

- With branch cuts (shown below)



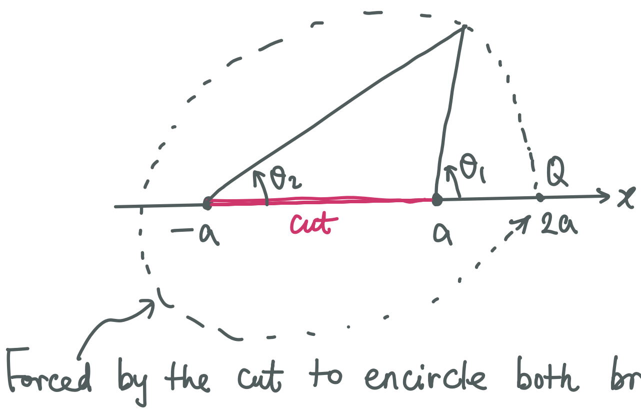
$C_Q \to \theta_1 = \theta_2 = \pi$

$C_R \to \theta_1 = \pi, \theta_2 = -\pi$

$$\to e^{i\frac{\theta_1+\theta_2}{2}} = \begin{cases} e^{i\pi} = -1 & \text{for } Q \\ e^{0} = +1 & \text{for } R \end{cases}$$

Discontinuity!

$\therefore$ For center cracks, may take a finite branch cut below



Initially : $r_1 = a$, $r_2 = 3a$, $\theta_1 = \theta_2 = 0$

Finally : $r_1 = a$, $r_2 = 3a$, $\theta_1 = \theta_2 = 2\pi$

$\rightarrow e^{\frac{\theta_1+\theta_2}{2}} \equiv 1$ ✓

Note that : ① This branch cut leads to discontinuity across $|x|<a, y=0$, (say $\theta_1 = \pi$, $\theta_2 = 0 \rightarrow \theta_1 = \pi$, $\theta_2 = 2\pi$ by cricling). This is fine physically as we have discontinuity across a crack.

② The cut does <u>not</u> render θ_1, θ_2 single valued (e.g., at Q we have $\theta_1 = \theta_2 = 2n\pi$, $n = 0, 1, \cdots$), but it is still a suitable cut since it renders <u>function</u> single valued (All that we ask!).

Anti-plane anisotropic crack tip fields

We have presumed the asymptotic form $Z_I \sim (2\pi r)^{-1/2}$. Does this work for more general anisotropic elasticity. Examine this for Mode III.

Equilibrium: $$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0$$

Kinematics: $$\gamma_{xz} = \frac{\partial w}{\partial x}, \quad \gamma_{yz} = \frac{\partial w}{\partial y}$$

Material law: $$\sigma_{xz} = \mu_{xx}\gamma_{xz} + \mu_{xy}\gamma_{yz}$$

$$\sigma_{yz} = \mu_{xy}\gamma_{xz} + \mu_{yy}\gamma_{yz}$$

Note that $$\begin{bmatrix} \mu_{xx} & \mu_{xy} \\ \mu_{xy} & \mu_{yy} \end{bmatrix} = \begin{bmatrix} C_{55} & C_{45} \\ C_{45} & C_{44} \end{bmatrix}$$ ← Voight notation

Kinematics → Material laws → Equilibrium:

$$\mu_{xx}\frac{\partial^2 w}{\partial x^2} + 2\mu_{xy}\frac{\partial^2 w}{\partial x \partial y} + \mu_{yy}\frac{\partial^2 w}{\partial y^2} = 0$$

Change of variables: $z = x + py$, $\bar{z} = x + \bar{p}y$. ← to be specified.

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \\ \frac{\partial}{\partial y} &= p\frac{\partial}{\partial z} + \bar{p}\frac{\partial}{\partial \bar{z}} \end{aligned}\right\} \rightarrow \begin{aligned} \frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2} \\ \frac{\partial^2}{\partial y^2} &= p^2\frac{\partial^2}{\partial z^2} + 2p\bar{p}\frac{\partial^2}{\partial z \partial \bar{z}} + \bar{p}^2\frac{\partial^2}{\partial \bar{z}^2} \end{aligned}$$

$$\frac{\partial^2}{\partial x \partial y} = p\frac{\partial^2}{\partial z^2} + (p+\bar{p})\frac{\partial^2}{\partial z \partial \bar{z}} + \bar{p}\frac{\partial^2}{\partial \bar{z}^2}$$

$$\rightarrow \frac{\partial^2}{\partial z^2}\left[\mu_{xx} + 2\mu_{xy}p + \mu_{yy}p^2\right]w + \frac{\partial^2}{\partial z \partial \bar{z}}\left[2\mu_{xx} + 2\mu_{xy}(p+\bar{p}) + 2\mu_{yy}p\bar{p}\right]w$$

$$+ \frac{\partial^2}{\partial \bar{z}^2}\left[\mu_{xx} + 2\mu_{xy}\bar{p} + \mu_{yy}\bar{p}^2\right]w = 0$$

Take $\mu_{xx} + 2\mu_{xy}p + \mu_{yy}p^2 = 0$ to get ride of $\frac{\partial^2}{\partial z^2}$ and $\frac{\partial^2}{\partial \bar{z}^2}$ terms:

$$\rightarrow p = \frac{-2\mu_{xy} \pm \sqrt{4\mu_{xy}^2 - 4\mu_{xx}\mu_{yy}}}{2\mu_{yy}}$$

Note that strain energy for any $\gamma_{xz}, \gamma_{yz} \geq 0$ requirs $\underbrace{\underset{\approx}{\mu} \text{ positively definite}}_{\mu_{xx}>0,\ \mu_{yy}>0,\ \mu_{xx}\mu_{yy}-\mu_{xy}^2>0}$

$$\rightarrow p = \underbrace{-\frac{\mu_{xy}}{\mu_{yy}}}_{p_r} \pm i \underbrace{\frac{\sqrt{\mu_{xx}\mu_{yy} - \mu_{xy}^2}}{\mu_{yy}}}_{p_i} \rightarrow \begin{cases} p = p_r + ip_i \\ \bar{p} = p_r - ip_i \end{cases}, \quad \text{Let } \boxed{\mu = \sqrt{\mu_{xx}\mu_{yy} - \mu_{xy}^2}}$$

Then, we have $\frac{\partial^2 w}{\partial z \partial \bar{z}} = 0 \rightarrow W = F(z) + G(\bar{z}) = F(z) + \underbrace{\overline{F(z)}}_{\text{So that } W \text{ is real}} = 2Re\left[F(z)\right]$

$$\begin{cases} \gamma_{xz} = \frac{\partial W}{\partial x} = F'(z) + \overline{F'(z)} = 2Re\left[F'(z)\right], \quad \gamma_{yz} = \frac{\partial W}{\partial y} = pF'(z) + \overline{pF'(z)} = 2Re\left[pF'(z)\right] \\ \sigma_{xz} = \mu_{xx}(F' + \overline{F'}) + \mu_{xy}(pF' + \overline{pF'}), \quad \sigma_{yz} = \mu_{xy}(F' + \overline{F'}) + \mu_{yy}(pF' + \overline{pF'}) \end{cases}$$

Note that

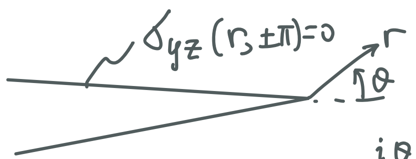
$$\mu_{xy} + \mu_{yy} p = i\, p_i\, \mu_{yy} \quad \longrightarrow \quad \frac{\mu_{xy} + \mu_{yy} p}{\mu} = i$$

$$\mu_{xx} + \mu_{xy} p = \frac{\mu^2}{\mu_{yy}} + i\, p_i \cdot \mu_{xy} \quad \longrightarrow \quad \frac{\mu_{xx} + \mu_{xy} p}{\mu} = \frac{\mu}{\mu_{yy}} + i\,\frac{\mu_{xy}}{\mu_{yy}} = -ip$$

$$\rightarrow \quad \sigma_{xz} = \mu\left(-ipF' - \overline{ipF'}\right) = 2\mu\, \mathrm{Re}\left[-ipF'\right] = 2\mu\, \mathrm{Im}\left[pF'(z)\right]$$

$$\sigma_{yz} = \mu\left(iF' + \overline{iF'}\right) = 2\mu\, \mathrm{Re}\left[iF'\right] = -2\mu\, \mathrm{Im}\left[F'(z)\right]$$

Now, consider the crack solution



$z = re^{i\theta}$

Try $F'(z) = Az^s = (A_r + iA_i)\, r^s e^{is\theta}$

$$\sigma_{yz}(r, \pm\pi) = -2\mu\, \mathrm{Im}\left[(A_r + iA_i)\, r^s \left(\cos(\pm s\pi) + i\sin(\pm s\pi)\right)\right]$$

$$= -2\mu r^s \left[\pm A_r \sin(s\pi) + A_i \cos(s\pi)\right] \equiv 0$$

$$\rightarrow s = \frac{n}{2} \;(n \in \text{odd}) \text{ and } A_r = 0 \quad \text{or} \quad s = n \;(n \in I) \text{ and } A_i = 0$$

The arguement of finite energy requires $s > -1$. The most singular term is given by $s = -\frac{1}{2}$, i.e., $A_r = 0$:

$$F'(z) = i\,\frac{A_i}{z^{1/2}}$$

Irwin's normalization: $\sigma_{yz}(r, \theta = 0) = \dfrac{K_{III}}{\sqrt{2\pi r}} \underset{\text{on } \theta = 0}{\overset{z = r}{=\!=\!=}} -2\mu \cdot \dfrac{A_i}{\sqrt{r}} \rightarrow A_i = -\dfrac{K_{III}}{2\sqrt{2\pi}\,\mu}$

$$\rightarrow F'(z) = -\frac{iK_{III}}{2\mu\sqrt{2\pi z}}, \quad F(z) = -\frac{iK_{III}}{\mu}\sqrt{\frac{z}{2\pi}} + z_0$$

The tearing displacement $\delta = w(r,\pi) - w(r,-\pi)$

$$= 4\,\mathrm{Re}\left[-\frac{iK_{III}}{\mu}\sqrt{\frac{re^{i\pi}}{2\pi}}\right]$$

$$= \frac{4K_{III}}{\mu}\sqrt{\frac{r}{2\pi}}$$

$$\rightarrow \mathcal{G}\,\delta a = \underbrace{\frac{1}{2}\int_0^{\delta a} \frac{K_{III}}{\sqrt{2\pi r}} \cdot \frac{4K_{III}}{\mu}\sqrt{\frac{\delta a - r}{2\pi}}\, dr}_{\text{crack closure integral}} = \frac{K_{III}^2}{2\mu}\,\delta a$$

$$\therefore \boxed{\mathcal{G} = \frac{K_{III}^2}{2\mu}, \quad \mu = \sqrt{\mu_{xx}\mu_{yy} - \mu_{xy}^2}}$$