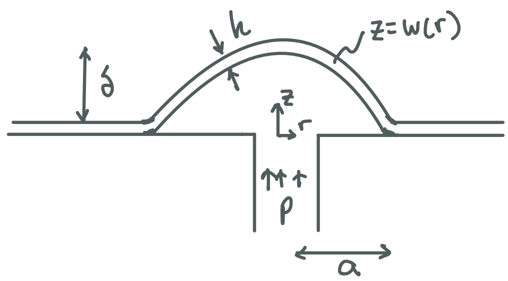
Circular pressurized bulge



Let's still focus on linear material law and moderate rotation.

In the axisymmetric configuration, there's a pair of equilibrium equations (see Mansfield, 2005)

Out of plane equilibrium equation:

$$\underbrace{\nabla^2(B\nabla^2 w)}_{\text{Bending}} - \underbrace{(N_{rr}K_{rr} + N_{\theta\theta}K_{\theta\theta})}_{\text{Stretching}} = p \,, \quad \underbrace{K_{rr} = \frac{d^2w}{dr^2} \,, \quad K_{\theta\theta} = \frac{1}{r}\frac{dw}{dr}}_{\text{Curvatures}}$$

In plane equilibrium equation:

$$\frac{dN_{rr}}{dr} + \frac{N_{rr} - N_{\theta\theta}}{r} = 0$$

Material law: $\varepsilon_{rr} = \frac{1}{Eh}(N_{rr} - \nu N_{\theta\theta})$, $\varepsilon_{\theta\theta} = \frac{1}{Eh}(N_{\theta\theta} - \nu N_{rr})$

or

$$N_{rr} = \frac{Eh}{1-\nu^2}(\varepsilon_{rr} + \nu\,\varepsilon_{\theta\theta}) \,, \quad N_{\theta\theta} = \frac{Eh}{1-\nu^2}(\varepsilon_{\theta\theta} + \nu\,\varepsilon_{rr})$$

Kinematic relations: $\varepsilon_{rr} = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2$, $\varepsilon_{\theta\theta} = \frac{u}{r}$

Boundary conditions: $\frac{dw}{dr} = \frac{d^3w}{dr^3} = 0$, $u = 0$ at $r = 0$ (symmetry)

$w = \frac{dw}{dr} = 0$, $u = \frac{N_m(1-\nu)}{Eh}$ at $r = a$

Residual stress.

Energy release rate: $\mathcal{G} = -\frac{\partial \Pi}{\partial A}$ ("global")

$= \frac{1-\nu^2}{2Eh}\Delta N_a^2 + \frac{1}{2B}M_a^2$ ("local")

Let's examine the elastic energy (density) associated with bending, induced tension and residual tension.

- $U_b \sim B\kappa^2 \sim B\left(\frac{\delta}{a^2}\right)^2 \sim B\frac{\delta^2}{a^4}$ (Bending)
- $U_s \sim Eh\varepsilon_s^2 \sim Eh\left(\frac{\delta}{a}\right)^4 \sim Eh\frac{\delta^4}{a^2}$ (Induced tension)
- $U_p \sim N_m\varepsilon_s \sim N_m\left(\frac{\delta}{a}\right)^2 \sim N_m\frac{\delta^2}{a^2}$ (Residual tension)

The system can be linearized as long as U_s is not important since only this term involves nonlinear kinematics. For example, when $U_p \gg U_s$, i.e., $Eh\frac{\delta^2}{a^2} \ll N_m$, both N_{rr} and $N_{\theta\theta}$ approach N_m. The in-plane equation is satisfied automatically, and the out of plane equation becomes

$$B\nabla^4 w - N_m\nabla^2 w = p. \qquad (*)$$

The solution can be readily obtained: $w(r) = \underbrace{\frac{pa^2}{4N_m}\left(1-\frac{r^2}{a^2}\right)}_{\text{Particular sol.}} + w_h(r)$

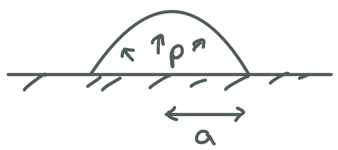
To seek w_h, note that the solution $\nabla^2 w - \lambda w = 0$ is $w = C_1 + C_2\log r$ for $\lambda = 0$ and $w = C_3 I_0(\sqrt{\lambda}\,r) + C_4 K_0(\sqrt{\lambda}\,r)$ where I_0 and K_0 are modified Bessel functions of the first and the second kind of order 0. Seek solution of $(*)$ in the form of $\nabla^2 w = \lambda w$,

$$\lambda^2 - \frac{N_m}{B}\lambda = 0 \rightarrow \lambda_1 = 0,\ \lambda_2 = \frac{N_m}{B} \rightarrow w_h(r) = C_1 + C_2\log r + C_3 I_0\left(\sqrt{\frac{N_m}{B}}\,r\right) + \underbrace{C_4 K_0\left(\sqrt{\frac{N_m}{B}}\,r\right)}_{\sim -\log r \text{ as } r\to 0}$$

With BCs, you'll be able to figure out C_1, C_2, C_3, C_4 and $G = G(p)$ or $G(\delta)$.

It has been shown that as $N_m a^2 \ll B$, $w \rightarrow \frac{pa^4}{64B}\left(1-\frac{r^2}{a}\right)^2$, $G \rightarrow \frac{p^2a^4}{128B}$, $\psi \rightarrow -45°$.
(Freund & Suresh, 2003)

Finally, let's discuss the membrane response with $N_m = 0$. Once again, start with scalings.



$$\varepsilon \sim \delta^2/a^2 \longrightarrow \varepsilon_0 = \psi(\nu)\, \delta^2/a^2 \quad \text{strain level at the center.}$$

$$\underbrace{p \times a^2 \cdot \delta}_{\sim \text{Volume}} \sim Eh\varepsilon^2 \times \underbrace{a^2}_{\sim \text{Area}} \rightarrow p = \xi(\nu)\, Eh \frac{\delta^3}{a^4}$$

$$G \sim \frac{\Pi}{a^2} \sim Eh\varepsilon^2 \rightarrow G = \phi(\nu)\, Eh \frac{\delta^4}{a^4}$$

$$G \sim \frac{p a^2 \delta}{a^2} \longrightarrow G = \left[\varphi(\nu) \frac{p^4 a^4}{Eh}\right]^{1/3}$$

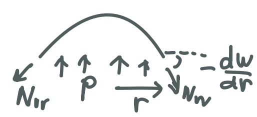
Need to solve the boundary value problem:

$$N_{rr} \frac{d^2 w}{dr^2} + N_{\theta\theta} \frac{1}{r}\frac{dw}{dr} + p = 0 \quad (1) \qquad \& \qquad \frac{dN_{rr}}{dr} + \frac{N_{rr} - N_{\theta\theta}}{r} = 0 \quad (2)$$

subject to $u(0) = 0$, $w'(0) = 0$, $w(a) = 0$, $u(a) = 0 \leftarrow$ Neglected N_m.

· **Hencky's solution (1915)** (see NASA Techical Report L-17585)

$$(2) \rightarrow \frac{d(rN_{rr})}{dr} - N_{\theta\theta} = 0$$



$$(1)+(2) \rightarrow N_{rr} \frac{d^2 w}{dr^2} + \frac{d(rN_{rr})}{dr} \frac{1}{r}\frac{dw}{dr} = -p \rightarrow \frac{d}{dr}\left(r N_{rr} \frac{dw}{dr}\right) = -pr \rightarrow \boxed{N_{rr}\frac{dw}{dr} = -\frac{1}{2}pr} \quad (3)$$

$$\varepsilon_r = \frac{d(r\varepsilon_\theta)}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \rightarrow \frac{1}{2}Eh\left(\frac{dw}{dr}\right)^2 = N_{rr} - \nu N_{\theta\theta} - \frac{d}{dr}\left[r(N_{\theta\theta} - \nu N_{rr})\right]$$

$$(2) \rightarrow = N_{rr} - \cancel{\nu \frac{d(rN_{rr})}{dr}} - \frac{d}{dr}\left[r \frac{d}{dr}(rN_{rr}) - \cancel{\nu r N_{rr}}\right]$$

$$\rightarrow \boxed{r\frac{d}{dr}\left[\frac{d}{dr}(rN_{rr}) + N_{rr}\right] + \frac{1}{2}Eh\left(\frac{dw}{dr}\right)^2 = 0} \quad (4)$$

$$(3)+(4) \rightarrow \boxed{N_{rr}^2 \frac{d}{dr}\left[\frac{d}{dr}(rN_{rr}) + N_{rr}\right] + \frac{1}{8}Eh\,p^2 r = 0} \quad (5)$$

Hencky solved this problem by assuming the following form:

$$N_{rr} = \frac{1}{4}\left(Ehp^2a^2\right)^{1/3} \sum_0^\infty b_{2n}\left(\frac{r}{a}\right)^{2n}, \quad w = \left(\frac{pa^4}{Eh}\right)^{1/3} \sum_0^\infty a_{2n}\left[1-\left(\frac{r}{a}\right)^{2n+2}\right]$$

$$\xrightarrow{②} \quad N_{\theta\theta} = \frac{1}{4}\left(Ehp^2a^2\right)^{1/3} \sum_0^\infty (2n+1)\, b_{2n}\underbrace{\left(\frac{r}{a}\right)}_{=\rho}{}^{2n}$$

Plugging these into ⑤ & ③ gives

$$\left(b_0+b_2\rho^2+b_4\rho^4+\cdots\right)^2\left(8b_2\rho+24b_4\rho^3+48b_6\rho^5+\cdots\right) = -8\rho$$

$$\rightarrow \left(b_0^2b_2+1\right)\rho + \left(3b_0^2b_4+2b_0b_2^2\right)\rho^3 + \left(6b_0^2b_6+6b_0b_2b_4+b_2\left(b_2^2+2b_0b_4\right)\right)\rho^5+\cdots = 0$$

$$\therefore\ b_2 = -1/b_0^2,\quad b_2 = -2/(3b_0^5),\quad b_6 = -13/(18b_0^8),\ \cdots,\ b_{14} = -219241/(63504\, b_0^{20})$$

$$\left(b_0+b_2\rho^2+b_4\rho^4+\cdots\right)\left(a_0+2a_2\rho^2+3a_4\rho^4+4a_6\rho^6+\cdots\right) = 1$$

$$\rightarrow \left(b_0a_0-1\right) + \left(2b_0a_2+b_2a_0\right)\rho^2 + \left(3b_0a_4+2b_2a_2+b_4a_0\right)\rho^4+\cdots = 0$$

$$\therefore\ a_0 = 1/b_0,\quad a_2 = 1/(2b_0^4),\quad a_4 = 5/(9b_0^7),\ \cdots,\ a_{12} = 17051/(5292\, b_0^{19})$$

Now the only unknown is b_0, to be determined with boundary conditions. Note that $w'(0) = \underbrace{u(0)}_{=\lim_{r\to 0} r\varepsilon_\theta} = 0$, $w(r=a)=0$ has been satisfied in the assumed form of N_r & w. The "unused" condition comes from $u(r=a) = \frac{N_{rr}(1-\nu)}{Eh}\cdot a = 0$ since $N_{rr}=0$, i.e.,

$$u(r=a) = r\cdot\frac{1}{Eh}\left(N_{\theta\theta}-\nu N_{rr}\right)\Big|_{r=a} = r\cdot\frac{1}{Eh}\left[\frac{d(rN_{rr})}{dr}-\nu N_{rr}\right]_{r=a} = 0$$

$$\rightarrow (1-\nu)\,b_0 - (3-\nu)\frac{1}{b_0^2} - (5-\nu)\frac{1}{3b_0^5} + \cdots + (15-\nu)\frac{219241}{63504}\frac{1}{b_0^{20}} = 0$$

$\nu = 0.2$, $b_0 = 1.6827$; $\nu = 0.3$, $b_0 = 1.7244$

Jensen (1991): $\delta = \left[\varphi(\nu)\,\frac{p^4a^4}{Eh}\right]^{1/3}$, $\varphi(\nu=0.5) \approx 0.0523$

Komaragiri et at (2005): $p = \xi(\nu)\,Eh\,\frac{\delta^3}{a^4}$, $\xi(\nu) = \left(0.7179-0.1706\nu-0.1495\nu^2\right)^{-3}$.

• Simplified Kinematics

Williams (1997), Wan & Lim (1998), Freund & Suresh (2004)

Yue et al. (2012), Dai et al. (2018)

The idea is to assume kinematically admissible deformation fields and then determine the unknown coefficients using the principle of minimum potential energy. For example, a simple, two-parameter form has been used:

$$\underbrace{w(r) = \delta\left(1-\frac{r^2}{a^2}\right)}_{\text{Satisfy } w'(0)=w(a)=0} \quad \& \quad \underbrace{u(r) = u_0\frac{r}{a}\left(1-\frac{r}{a}\right)}_{\text{Satisfy } u(0)=u(a)=0}$$

Then, radial and hoop strain components can be calculated immediately

$$\varepsilon_{rr} = \frac{u_0}{a}\left(1-2\frac{r}{a}\right) + 2\frac{\delta^2 r^2}{a^4}, \quad \varepsilon_{\theta\theta} = \frac{u_0}{a}\left(1-\frac{r}{a}\right)$$

The elastic strain energy (per unit area) is

$$U(r) = \frac{Eh}{2(1-\nu^2)}\left(\varepsilon_{rr}^2 + 2\nu\,\varepsilon_{rr}\varepsilon_{\theta\theta} + \varepsilon_{\theta\theta}^2\right)$$

The total potential energy can be calculated as

$$\Pi(u_0, \delta) = 2\pi\int_0^a U(r)\,r\,dr - 2\pi p\int_0^a w(r)\,dr$$

The relation between p and δ and u_0 can be obtained by solving

$$\frac{\partial\Pi}{\partial u_0} = 0 \quad \& \quad \frac{\partial\Pi}{\partial\delta} = 0$$

I will not show the results here since the accuracy given by this method is not satisfactory. Dai et al PRL (2018) showed using $u(r) = u_0\frac{r}{a}\left(1-\frac{r^2}{a^2}\right)$ can help slightly. Further improving the accuracy needs more terms in the assumed kinematics but would lose its advantages in simplicity (compared to Hencky).

• Perturbed spherical cap shapes   Dai JAM (2024)

The idea is that the shape of the bulge is not a spherical cap exactly. But it appears quite close. Naturally seek solution around a parabola.

$$w(r) = \begin{cases} \delta\left[1-\left(\frac{r}{a}\right)^{2+\alpha}\right], & |\alpha| \ll 1 \quad \text{Solution I} \\ \delta\left[1-\left(\frac{r}{a}\right)^{2}\right]^{\beta}, & |\beta| \ll 1 \quad \text{Solution II} \\ \frac{\delta}{1+\epsilon}\left[1-\left(\frac{r}{a}\right)^{2}\right] + \frac{\epsilon\delta}{1+\epsilon}\left[1-\left(\frac{r}{a}\right)^{N}\right], & |\epsilon| \ll 1 \quad \text{Solution III} \end{cases}$$

Regarding the in-plane displacement field, instead of assuming a kinematically admissible form, one can directly solve for it based on in-plane equilibrium equation in terms of displacements:

$$\frac{d}{dr}(r N_{rr}) - N_{\theta\theta} = 0 \rightarrow \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \underbrace{\frac{1-\nu}{2r}\left(\frac{dw}{dr}\right)^2 + \frac{dw}{dr}\frac{d^2w}{dr^2}}_{\text{The nonlinear term now is explicit.}} = 0$$

For example, plugging $w(r)$ in solution I, together with $u(0)=u(a)=0$, can give

$$u(r) = \frac{(2+\alpha)(3+2\alpha-\nu)}{8(1+\alpha)}\delta^2 \frac{r}{a}\left[1-\left(\frac{r}{a}\right)^{2+2\alpha}\right]$$

Then we can use kinematic relations to calculate ε_{rr}, $\varepsilon_{\theta\theta}$ and $\Pi(\delta,\alpha)$. Finally using principle of minimum potential energy $\frac{\partial \Pi}{\partial \delta} = \frac{\partial \Pi}{\partial \alpha} = 0$ as well as $|\alpha| \ll 1$ leads to

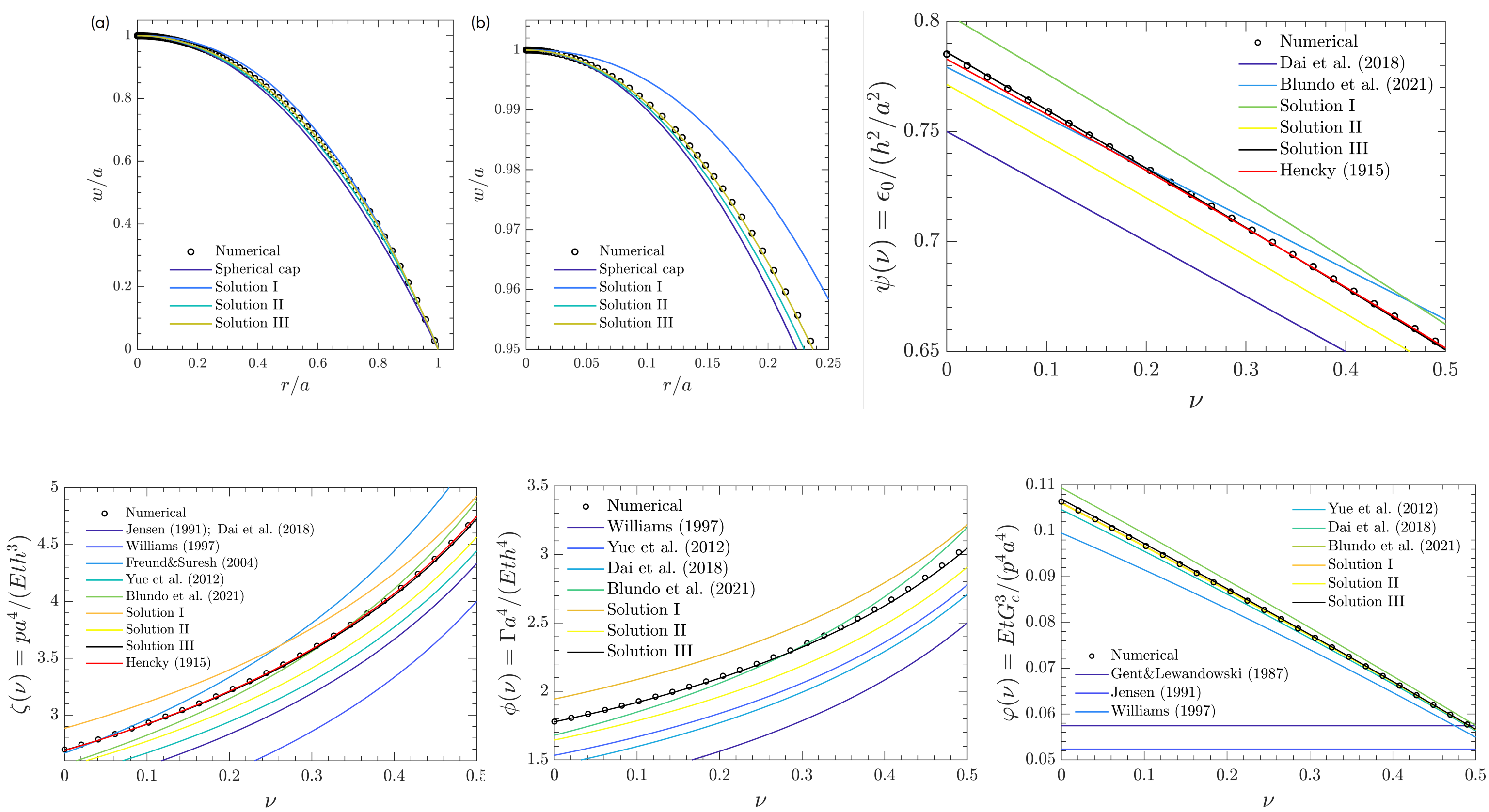
$$\alpha^{I}(\nu) \approx \frac{\sqrt{1025-742\nu+41\nu^2} - 15 - 3\nu}{50-2\nu}, \quad \delta^{I} = \delta^{I}(\nu), \quad \phi^{I} = \phi^{I}(\nu), \quad \varphi^{I} = \varphi^{I}(\nu).$$

based on solution I

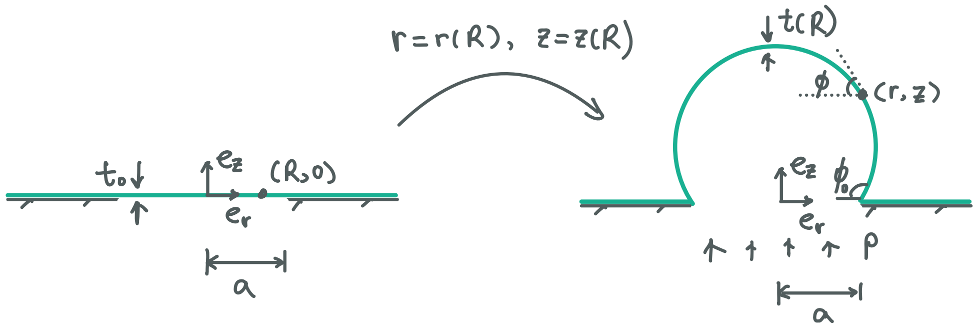
Similarly, these parameters can be calculated by using solution II & III. It is found Solution III with $N=5$ works particularly well. Specifically,

$$
\begin{cases}
w(r) = \dfrac{\delta}{1+\epsilon}\left(1-\dfrac{r^2}{a^2}\right) + \dfrac{\epsilon\delta}{1+\epsilon}\left(1-\dfrac{r^5}{a^5}\right), \quad \epsilon \simeq \dfrac{987-231\nu-7\sqrt{10985+3878\nu-3199\nu^2}}{12(139-67\nu)} \\[2ex]
\varepsilon_0 = \psi(\nu)\,\delta^2/a^2, \qquad \psi(\nu) = \dfrac{3-\nu}{4} + \dfrac{3(1+\nu)}{14}\epsilon + O(\epsilon^2) \\[2ex]
p = \zeta(\nu)\, Eh\dfrac{\delta^3}{a^4}, \qquad \zeta(\nu) = \dfrac{7-\nu}{3(1-\nu)} + \dfrac{149+13\nu}{63(1-\nu)}\epsilon + O(\epsilon^2) \\[2ex]
G = \phi(\nu)\, Eh\dfrac{\delta^4}{a^4}, \qquad \phi(\nu) = \dfrac{5(7-\nu)}{24(1-\nu)} + \dfrac{5(53+\nu)}{126(1-\nu)}\epsilon + O(\epsilon^2) \\[2ex]
G = \left[\varphi(\nu)\dfrac{p^4a^4}{Eh}\right]^{1/3}, \qquad \varphi(\nu) = \dfrac{375(1-\nu)}{512(7-\nu)} + \dfrac{625(1-\nu)^2}{448(7-\nu)^2}\epsilon + O(\epsilon^2)
\end{cases}
$$

Since $\epsilon \sim 0.1$ for typical ν, this solution reduces the errors within $\epsilon^2 \sim 1\%$.



Circular pressurized hyperelastic bulge



λ_r, λ_θ, λ_h denote principal stretches of the membrane along the radial, hoop, and thickness directions.

$$\lambda_r = \sqrt{r'^2 + z'^2}, \quad \lambda_\theta = \frac{r}{R}, \quad \lambda_t = \frac{t}{t_0}$$

current thickness (t), initial thickness (t_0).

We assume the film is incompressible so that $\lambda_r \lambda_\theta \lambda_t \equiv 1$, i.e., $t = t_0/(\lambda_r \lambda_\theta)$

The volume of the bulge is $V = \int_0^a 2\pi r z \, dr = \int_0^a 2\pi r r' z \, dR$

The total potential energy can be written as

$$\Pi = \underset{\text{strain energy}}{U_m} - pV$$

$$= \int_0^a W t_0 2\pi R \, dR - p \int_0^a 2\pi r r' z \, dR$$

$$= 2\pi \int_0^a \underbrace{\left(W t_0 R - p r r' z \right)}_{= F(r, r', z, z')} dR$$

where $W = W(\lambda_r, \lambda_\theta)$ is the strain energy per unit volume in the undeformed configuration.

Therefore $\Pi = \Pi(r, r', z, z')$. Let's then examine $\delta\Pi$ with $\delta R_0 \neq 0$ since we want to know $G = -\delta\Pi/\delta(\pi R_0^2)$.

$$\delta \int_0^a F \, dR = \int_0^a \left(\frac{\partial F}{\partial r} \delta r + \frac{\partial F}{\partial r'} \delta r' + \frac{\partial F}{\partial z} \delta z + \frac{\partial F}{\partial z'} \delta z' \right) dR + F\Big|_a \delta a$$

Note that $\delta \int_0^a \frac{\partial F}{\partial r'} \delta r' \, dR = \frac{\partial F}{\partial r'} \delta r \Big|_a - \frac{\partial F}{\partial r'} \delta r \Big|_0 - \int_0^a \frac{d}{dR}\left(\frac{\partial F}{\partial r'} \right) \delta r \, dR$

$$\delta \int_0^a \frac{\partial F}{\partial z'} \delta z' dR = \frac{\partial F}{\partial z'} \delta z \Big|_a - \frac{\partial F}{\partial z'} \delta z \Big|_0 - \int_0^a \frac{d}{dR}\left(\frac{\partial F}{\partial z'}\right) \delta z \, dR$$

$$\rightarrow \delta \Pi = \int_0^a \Big[\underbrace{\left(\frac{\partial F}{\partial r} - \frac{d}{dR}\frac{\partial F}{\partial r'}\right)}_{=0} \delta r + \underbrace{\left(\frac{\partial F}{\partial z} - \frac{d}{dR}\frac{\partial F}{\partial z'}\right)}_{=0} \delta z \Big] dR$$

$$+ \left(\frac{\partial F}{\partial r'} \delta r \Big|_a + \frac{\partial F}{\partial z'} \delta z \Big|_a + F\Big|_a \delta a - \frac{\partial F}{\partial r'} \delta r \Big|_0 - \frac{\partial F}{\partial z'} \delta z \Big|_0 \right) \leftarrow \text{B.T.s.}$$

Let's first examine the two Lagrangians

$$\frac{\partial F}{\partial r} - \left(\frac{\partial F}{\partial r'}\right)' = \frac{\partial W}{\partial r} t_0 R - p r' z - \left(R \frac{\partial W}{\partial r'}\right)' t_0 + p(r'z + rz')$$

$$= t_0 \left[R \frac{\partial W}{\partial r} - \left(R \frac{\partial W}{\partial r'}\right)' \right] + p r z' = 0 \qquad ①$$

$$\frac{\partial F}{\partial z} - \left(\frac{\partial F}{\partial z'}\right)' = t_0 \left[R \frac{\partial W}{\partial z} - \left(R \frac{\partial W}{\partial z'}\right)' \right] - p r r' = 0 \qquad ②$$

W is a function of $\lambda_r, \lambda_\theta$. Need to establish the relation between $\frac{\partial W}{\partial r}$ and $\frac{\partial W}{\partial \lambda_r}$ etc.

$$\frac{\partial W}{\partial r} = \frac{\partial W}{\partial \lambda_r} \cancelto{0}{\frac{\partial \lambda_r}{\partial r}} + \frac{\partial W}{\partial \lambda_\theta} \cdot \frac{\partial \lambda_\theta}{\partial r} = \frac{1}{R} \frac{\partial W}{\partial \lambda_\theta}$$

$$\frac{\partial W}{\partial r'} = \frac{\partial W}{\partial \lambda_r} \cdot \frac{\partial \lambda_r}{\partial r'} + \frac{\partial W}{\partial \lambda_\theta} \cancelto{0}{\frac{\partial \lambda_\theta}{\partial r'}} = \frac{r'}{\sqrt{r'^2 + z'^2}} \cdot \frac{\partial W}{\partial r'} = \frac{r'}{\lambda_r} \frac{\partial W}{\partial \lambda_r}$$

$$\frac{\partial W}{\partial z} = 0, \quad \frac{\partial W}{\partial z'} = \frac{z'}{\lambda_r} \frac{\partial W}{\partial \lambda_r}$$

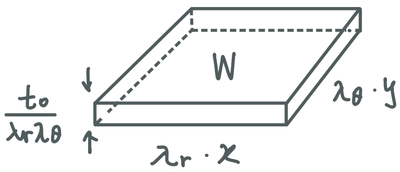
With these, we can rewrite ① and ② as

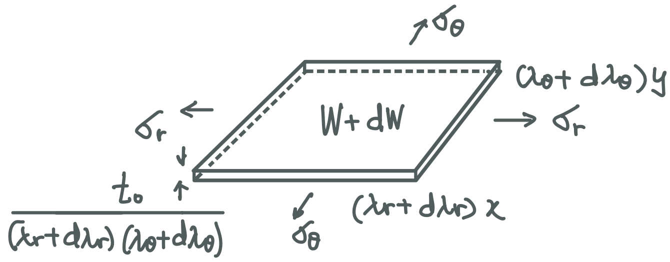
$$p = -\frac{t_0}{r z'} \left[\frac{\partial W}{\partial \lambda_\theta} - \frac{r'}{\lambda_r} \frac{\partial W}{\partial \lambda_r} - R \left(\frac{r'}{\lambda_r}\right)' \frac{\partial W}{\partial \lambda_r} - R \frac{r'}{\lambda_r} \left(\frac{\partial W}{\partial \lambda_r}\right)' \right]$$

③

$$p = \frac{t_0}{r r'} \left[- \frac{z'}{\lambda_r} \frac{\partial W}{\partial \lambda_r} - R \left(\frac{z'}{\lambda_r}\right)' \frac{\partial W}{\partial \lambda_r} - R \left(\frac{z'}{\lambda_r}\right) \left(\frac{\partial W}{\partial \lambda_r}\right)' \right]$$

What is the physical picture of $\frac{\partial W}{\partial \lambda_r}$, $\frac{\partial W}{\partial \lambda_\theta}$? We learnt $\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}$ so expect $\frac{\partial W}{\partial \lambda_r}$ and $\frac{\partial W}{\partial \lambda_\theta}$ will give rise to something like stress.





Stored energy: $\underbrace{W(\lambda_r, \lambda_\theta)}\, x y t_0$

defined in the undeformed configuration but it is the same in the deformed configuration when the material is incompressible:

$$\tilde{W}(\lambda_r, \lambda_\theta) \cdot \lambda_r x \cdot \lambda_\theta y \cdot \frac{t_0}{\lambda_r \lambda_\theta} = \tilde{W}\, x y t_0$$

Stored energy: $W(\lambda_r + d\lambda_r, \lambda_\theta + d\lambda_\theta)\, xy\, t_0$

$$= \left[W(\lambda_r, \lambda_\theta) + \frac{\partial W}{\partial \lambda_r} d\lambda_r + \frac{\partial W}{\partial \lambda_\theta} d\lambda_\theta \right] x y t_0$$

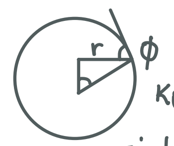
$$\overbrace{\underbrace{\sigma_r \cdot \lambda_\theta y \cdot \frac{t_0}{\lambda_r\lambda_\theta}}_{\text{Net force}} \cdot \underbrace{x \cdot d\lambda_r}_{\text{displc.}} + \sigma_\theta \lambda_r x \cdot \frac{t_0}{\lambda_r \lambda_\theta} \cdot y\, d\lambda_\theta}^{\text{External work}} = \overbrace{\left(\frac{\partial W}{\partial \lambda_r} d\lambda_r + \frac{\partial W}{\partial \lambda_\theta} d\lambda_\theta \right) x y t_0}^{\text{Change in stored energy}}$$

$$\rightarrow \left(\frac{\sigma_r}{\lambda_r} - \frac{\partial W}{\partial \lambda_r} \right) d\lambda_r + \left(\frac{\sigma_\theta}{\lambda_\theta} - \frac{\partial W}{\partial \lambda_\theta} \right) d\lambda_\theta = 0$$

$d\lambda_r$, $d\lambda_\theta$ are arbitary so that $\sigma_r = \lambda_r \frac{\partial W}{\partial \lambda_r}$, $\sigma_\theta = \lambda_\theta \frac{\partial W}{\partial \lambda_\theta}$. We are interested in stress resultants: $N_r = \sigma_r \cdot t = \frac{t_0}{\lambda_\theta} \frac{\partial W}{\partial \lambda_r}$, $N_\theta = \sigma_\theta t = \frac{t_0}{\lambda_r} \frac{\partial W}{\partial \lambda_\theta}$. Now rewrite ③ in terms of N_r, N_θ

$$p = -\frac{1}{r z'} \left[\lambda_r N_\theta - \frac{r'}{\lambda_r} \lambda_\theta N_r - R \left(\frac{r'}{\lambda_r} \right)' \cdot \lambda_\theta N_r - R \frac{r'}{\lambda_r} (\lambda_\theta N_r)' \right]$$

$$= -\frac{(r'^2 + z'^2)^{1/2}}{r z'} N_\theta + \frac{r'^4 + r'^2 z'^2 + r r'' z'^2 - r r' z z'}{r z' (r'^2 + z'^2)^{3/2}} N_r + \frac{r'}{z' (r'^2 + z'^2)^{1/2}} \frac{dN_r}{dR} \quad ④$$

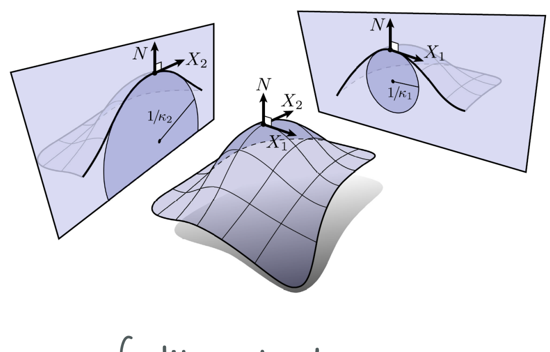


$\kappa_\theta = \frac{\sin\phi}{r}$

$\sin\phi = \frac{z'}{\lambda_r}$

$$p = \frac{1}{r r'} \left[-\frac{z'}{\lambda_r} \lambda_\theta N_r - R \left(\frac{z'}{\lambda_r} \right)' \lambda_\theta N_r - R \frac{z'}{\lambda_r} (\lambda_\theta N_r)' \right]$$

$$= -\frac{z'(r'^2 + z'^2 - r r'') + r r' z''}{r (r'^2 + z'^2)^{3/2}} N_r - \frac{z'}{r' (r'^2 + z'^2)^{1/2}} \frac{dN_r}{dR} \quad ⑤$$



$$\boxed{\begin{aligned} &④-⑤ \rightarrow \quad \frac{dN_r}{dR} + \frac{r'(N_r - N_\theta)}{r} = 0 \\ &④+⑤ \rightarrow \quad \underbrace{\frac{r' z'' - z' r''}{(r'^2 + z'^2)^{3/2}}}_{\kappa_r} N_r + \underbrace{\frac{z'}{r (r'^2 + z'^2)^{1/2}}}_{\kappa_\theta} N_\theta = p \end{aligned}}$$

$$\xrightarrow[r \to R,\ r' \to 1,\ |z'^2| \ll 1]{\text{Moderate rotation}} \left\{ \begin{aligned} &\frac{dN_r}{dR} + \frac{N_r - N_\theta}{R} = 0 \\ &N_{rr} z'' + N_{\theta\theta} \frac{z'}{r} = -p \end{aligned} \right.$$

(Foppel membrane)

Material law

Having given the equilibrium equations, we need to specify a material law to proceed. There are various types of material laws - we consider two of commonly used for soft materials.

Neo-Hookean model : $W = \frac{\mu}{2}\left(\underbrace{\lambda_r^2 + \lambda_\theta^2 + \frac{1}{\lambda_r^2\lambda_\theta^2}}_{=I_1} - 3\right)$

shear modulus of the membrane

Gent model : $W = -\frac{\mu}{2} J_m \log\left(1 - \frac{I_1 - 3}{J_m}\right)$

first invariant of Cauchy-Green tensor

i.e. $I_1 = \mathrm{tr}(\underset{\sim}{F}\underset{\sim}{F}^T)$

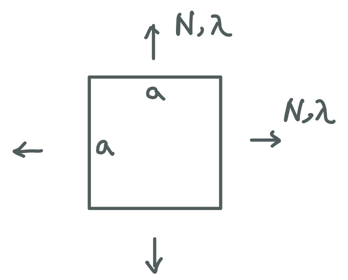
$= \lambda_r^2 + \lambda_\theta^2 + \frac{1}{\lambda_r^2\lambda_\theta^2}$

material constant

Note that as $J_m \gg 1$, Gent model behaves as $\frac{\mu}{2}\left(I_1 - 3 + O(J_m^{-1})\right)$. So we will try Gent model with a range of J_m

Aside .

Consider equibiaxial tension $\lambda_r = \lambda_\theta = \lambda$, $\lambda_h = 1/\lambda^2$

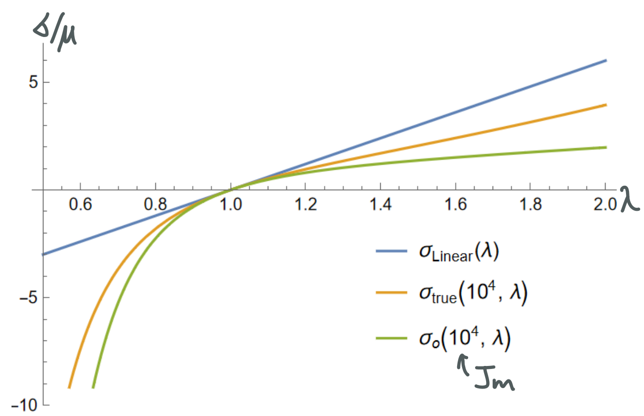


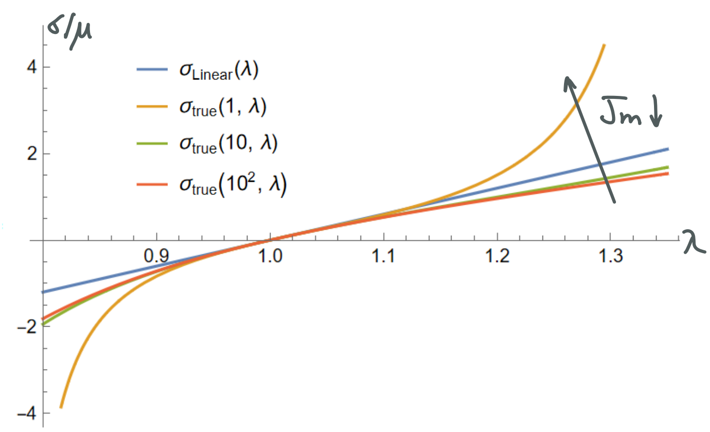
$$\underset{\text{true stress}}{\sigma} = \lambda_r \frac{\partial W}{\partial \lambda_r}\bigg|_{\lambda_r = \lambda_\theta = \lambda} = \frac{1}{2}\lambda\frac{\partial W}{\partial \lambda} = \mu J_m \frac{\lambda^6 - 1}{-2\lambda^6 + (3 + J_m)\lambda^4 - 1}$$

$$F = \sigma t a = \frac{t_0}{\lambda^2} a_0 \lambda \sigma$$

$$\rightarrow \underset{\text{nominal stress}}{\sigma_0} = \frac{F}{a_0 t_0} = \sigma/\lambda = \mu J_m \frac{\lambda^6 - 1}{-2\lambda^7 + (3 + J_m)\lambda^5 - \lambda}$$

Linear Hooke's law $\sigma = \sigma_0 = \frac{E}{1-\nu}\varepsilon = 6\mu(\lambda - 1)$ since $\mu = \frac{E}{2(1+\nu)} = \frac{E}{3}$ and $\nu = 0.5$





Using Gent material model, we obtain

$$N_r = \mu t_0 J_m \frac{\lambda_r^4 \lambda_\theta^2 - 1}{\lambda_r^3 \lambda_\theta^3 (J_m + 3 - \lambda_r^2 - \lambda_\theta^2) - \lambda_r \lambda_\theta}$$

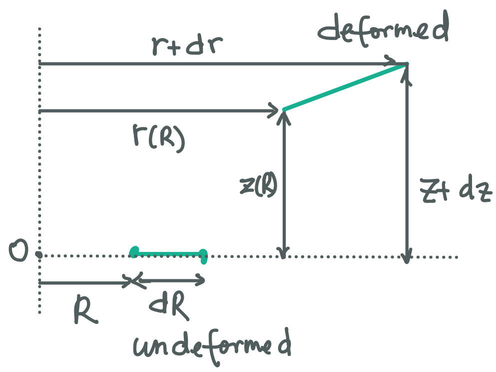
$$N_\theta = \mu t_0 J_m \frac{\lambda_r^2 \lambda_\theta^4 - 1}{\lambda_r^3 \lambda_\theta^3 (J_m + 3 - \lambda_r^2 - \lambda_\theta^2) - \lambda_r \lambda_\theta}$$

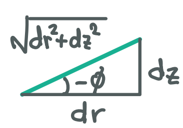
Numerics

Let's first solve the system with some natural boundary conditions

$$z'(0)=0, \quad z(a)=0, \quad r(0)=\lim_{R\to 0}(R+u(R))=0, \quad r(a)=\lim_{R\to a}(R+u(a))=a \quad \text{or } \lambda_\theta(a)=1$$

You'll find bvp solvers not quite efficient due to a good deal of nonlinearities. May try to solve a ivp problem using shooting method. The idea is to replace 2 second order coupled ODEs regarding r'', z'' with 4 first order ODEs. There are many options while we take $\lambda_\theta=\lambda_\theta(R)$, $\lambda_z=\lambda_z(R)$, $z=z(R)$, $\phi=\phi(R)$ here





$\rightarrow$

$$\boxed{\cos\phi = \frac{r'}{\lambda_r}, \quad \sin\phi = -\frac{z'}{\lambda_r} \qquad K_r = -\frac{\phi'}{\lambda_r}, \quad K_\theta = -\frac{\sin\phi}{r}}$$

$$-d\phi/ds = -\phi' \, dR/ds = -\phi'/\lambda_r$$

Need to know relations between λ_r', λ_θ', ϕ' and $\lambda_r, \lambda_\theta, \phi, z$

$$\lambda_\theta' = \frac{r'}{R} - \frac{r}{R^2} = \frac{\lambda_r \cos\phi}{R} - \frac{\lambda_\theta}{R} \checkmark$$

$$\frac{dN_r}{dR} + \frac{r'(N_r - N_\theta)}{r} = 0 \rightarrow \frac{dN_r}{d\lambda_\theta}\cdot\lambda_\theta' + \frac{dN_r}{d\lambda_r}\cdot\lambda_r' + \frac{r'(N_r - N_\theta)}{r} = 0$$

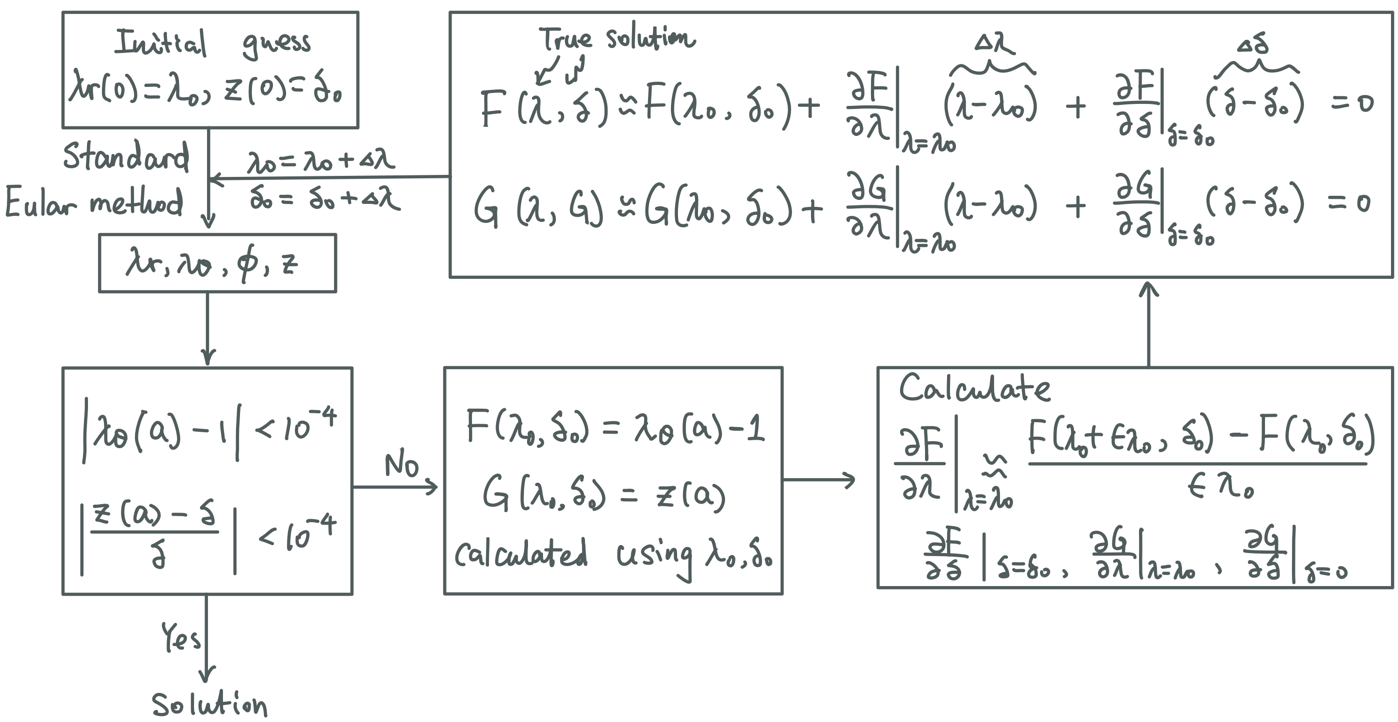
$$\rightarrow \lambda_r' = \left[-\frac{\lambda_r \cos\phi\,(N_r - N_\theta)}{\lambda_\theta R} - \frac{\lambda_r\cos\phi - \lambda_\theta}{R}\frac{dN_r}{d\lambda_\theta}\right] \Big/ \frac{dN_r}{d\lambda_r} \checkmark$$

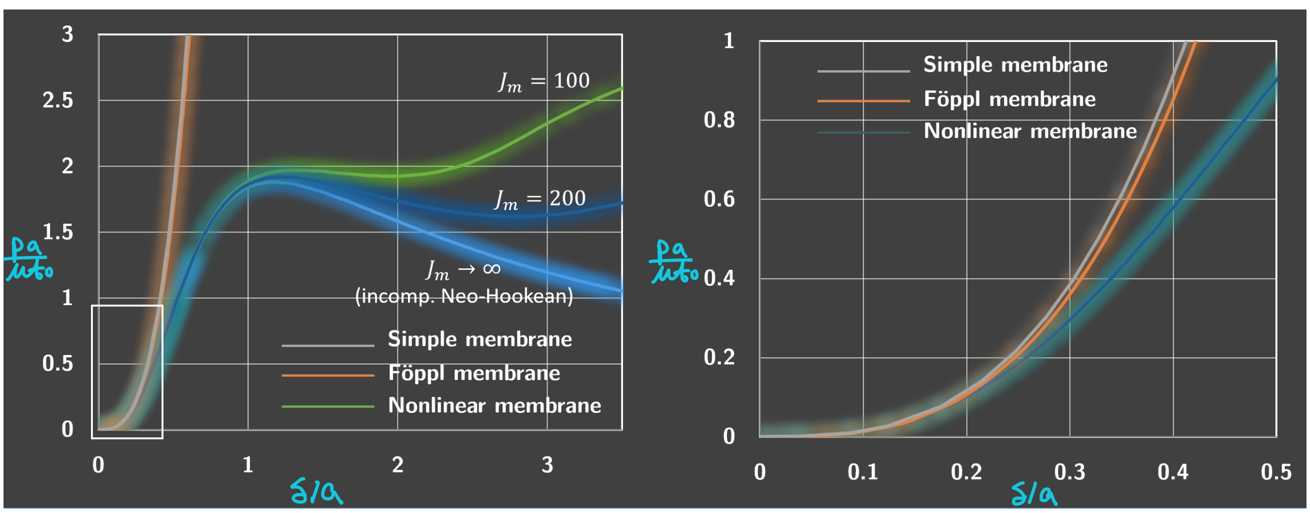
$$N_r K_r + N_\theta K_\theta = -p \rightarrow -\frac{\phi'}{\lambda_r}N_r - \frac{\sin\phi}{r}N_\theta = -p \rightarrow \phi' = \left(p - \frac{\sin\phi}{\lambda_\theta R}N_\theta\right)\lambda_r / N_r \checkmark$$

These equations complete the 4 ODEs: $\frac{dy_i}{dR} = f_i(y_i, R)$, which can be solved with initial conditions $y_i(0)$, given other parameters including μ, a, J_m, p:

$$\lambda_r(0) = \lambda_\theta(0) = \lambda, \quad \phi(0) = 0, \quad z(0) = \delta$$

However, λ and δ are not known a priori – the value of them should ensure that $\lambda_\theta = 1$ and $z = 0$ at $R = a$.





- Simple membrane: $N \nabla^2 w = -p$
- Föppl membrane: $N_{rr} K_{rr} + N_{\theta\theta} K_{\theta\theta} = -p$, $\frac{d(RN_r)}{dR} - N_\theta = 0$, $K_{rr}, K_{\theta\theta}$ linearized.
} Good only when $\left(\frac{\delta}{a}\right)^2 \ll 1$
- Nonlinear membrane: ① Snap through and stiffening for small J_m.

Energy release rate

The membrane-substrate interface toughness (say $\mathcal{G}_c$) is finite. Then interested in at which pressure the interface breaks, i.e., $\mathcal{G}(p) = \mathcal{G}_c$. How to calculate this? $\mathcal{G} = -\frac{\partial \Pi}{\partial (\pi a^2)}$

One immediate way is to compute Π_1 at given a and p and Π_2 at $a+\epsilon a$ with $0<\epsilon<<1$.

Then $\mathcal{G}(p) = -\lim_{\epsilon\to 0} \frac{\Pi_2 - \Pi_1}{2\pi a(\epsilon a)} = \lim_{\epsilon\to 0} \frac{\Pi_1 - \Pi_2}{2\pi a^2 \epsilon}$. The other way is to fing $\delta\Pi/(2\pi a \delta a)$ via variational analysis. Now revisit the boundary terms on Page 108:

$$\frac{\partial F}{\partial r'}\delta r\Big|_a + \frac{\partial F}{\partial z'}\delta z\Big|_a + F\Big|_a \delta a - \frac{\partial F}{\partial r'}\delta r\Big|_0 - \frac{\partial F}{\partial z'}\delta z\Big|_0$$

where $F(r, r', z, z') = W(r, r', z')\, t_0 R - p r r' z$.

P108 P109

$$\frac{\partial F}{\partial r'}\delta r = \left(\frac{\partial W}{\partial r'} t_0 R - p r z\right)\delta r = \left(\frac{r'}{\lambda_r}\frac{\partial W}{\partial \lambda_r} t_0 R - p r z\right)\delta r = \left(\frac{r' N_r \lambda_\theta}{\lambda_r} R - p r z\right)\delta r$$

$$\delta r|_0 = \delta r(0) = 0 \qquad \text{but } \delta r|_a = \delta r(a) - r'\big|_a \delta a = (1 - r'|_a)\delta a$$

$$\rightarrow \frac{\partial F}{\partial r'}\delta r\Big|_0 = 0, \quad \frac{\partial F}{\partial r'}\delta r\Big|_a = \left[\left(\frac{r' N_r \lambda_\theta}{\lambda_r} R - p r \cancelto{0}{z}\right)(1 - r')\right]_{R=a} \delta a \quad \leftarrow r' = \lambda_r \cos\phi$$

$$= N_r\big|_a \cos\phi_0 \cdot a \,(1 - \cos\phi_0 \lambda_r|_a)\,\delta a$$

$$\frac{\delta F}{\partial z'}\delta z = \left(\frac{\partial W}{\partial z'} t_0 R - p r r'\right)\delta z = \left(\frac{z'}{\lambda_r}\frac{\partial W}{\partial \lambda_r} t_0 R - p r \lambda_r \cos\phi\right)\delta z = \left(\frac{z' N_r \lambda_\theta}{\lambda_r} R - p r \lambda_r \cos\phi\right)\delta z$$

$$\delta z|_0 = \delta z(0), \quad \delta z|_a = \cancelto{0}{\delta z(a)} - z'|_a \delta a$$

$$\rightarrow \frac{\partial F}{\partial z'}\delta z\Big|_0 = 0\ \delta z(0) = 0, \quad \frac{\partial F}{\partial z'}\delta z\Big|_a = -\left(\frac{z' N_r \lambda_\theta}{\lambda_r} R - p r \lambda_r \cos\phi\right)\cdot z'\Big|_{R=a}\delta a \quad \leftarrow z' = -\lambda_r \sin\phi$$

$$= -\left(N_r|_a \sin\phi_0 \cdot a + p a \lambda_r|_a \cos\phi_0\right)\lambda_r|_a \sin\phi_0\, \delta a$$

$$F|_a \delta a = \left(W|_a t_0 a + p \cdot a \cdot \lambda_r^2|_a \cos\phi_0 \sin\phi\right)\delta a$$

$$\rightarrow \left(N_r|_a \cos\phi_0 a - (N_r\lambda_r)|_a \cos^2\phi_0\, a - (N_r\lambda_r)|_a \sin^2\phi_0\, a - p a \cancel{\lambda_r^2|_a \cos\phi_0 \sin\phi_0} + W|_a t_0 a + p a \cancel{\lambda_r^2|_a \cos\phi_0 \sin\phi_0}\right)\delta a$$

$$\rightarrow \mathcal{G} = -\frac{\partial \Pi}{\partial(\pi a^2)} = -\frac{\partial \Pi}{\underbrace{2\pi a}_{\text{cancelled out when using } \delta F}\partial a} = \left(N_r \lambda_r - N_r \cos\phi - t_0 W\right)_{R=a}$$

At small stretches, $\lambda_r \rightarrow 1+\varepsilon_r$, $W \rightarrow \frac{1}{2}N_r\varepsilon_r + \frac{1}{2}N_\theta \overset{\text{at } R=a}{\cancel{\varepsilon_\theta}}$

$$\mathcal{G} = \left[N_r(1-\cos\phi) + \frac{1}{2}N_r\varepsilon_r\right]_{R=a} ,$$

which returns to Kendall's peeling angle obtained using linear elasticity.