Thin film fracture

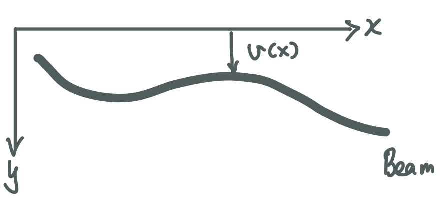
"Thin Film Materials" by L.B. Freund & S. Suresh: Thin films have been inserted into engineering systems in order to accomplish a wide range of practical sevice functions. Among these are micro (nano) electronic devices and packages, MEMS, and surface coating ··· ··· To a large extent, the success of this endeavor has been enabled by research leading to reliable means for estimating stress in small material systems and by eastablishing frameworks in which to access the integrity or functionality of the systems. → BVP for thin films!

The PDE

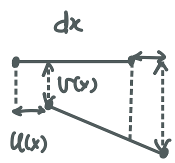
the criteria

Let us first consider a 2D case. We'll show many concepts obtained in 2D systems apply to more general 3D problems.

We consider partially nonlinear kinematics (i.e., moderate rotation) and linear material laws.



Displacement of neutral axis: $u(x)$, $v(x)$



$$u(x+dx) = u(x) + \frac{\partial u}{\partial x} dx + O(dx^2)$$
$$v(x+dx) = v(x) + \frac{\partial v}{\partial x} dx + O(dx^2)$$

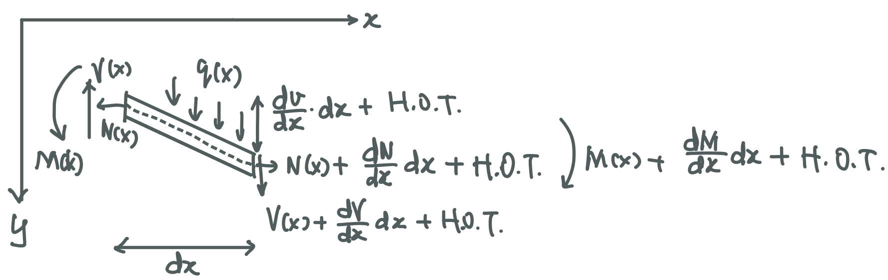
$$(dx')^2 = \left[dx + u(x+dx) - u(x)\right]^2 + \left[v(x+dx) - v(x)\right]^2$$
$$= \left(dx + \frac{\partial u}{\partial x} dx\right)^2 + \left(\frac{\partial v}{\partial x} dx\right)^2$$

$$\rightarrow dx' = dx\sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

2D-finite strain: $\varepsilon_{xx} = \dfrac{dx' - dx}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left[\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2\right]$

Small strain: $\frac{\partial u}{\partial x} \ll 1 \rightarrow \left(\frac{\partial u}{\partial x}\right)^2 \ll \frac{\partial u}{\partial x}$

Moderate rotation: $\left(\frac{\partial v}{\partial x}\right)^2 \sim \frac{\partial u}{\partial x} \ll 1 \rightarrow \boxed{\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2}$



$$\Sigma F_x = 0 \rightarrow \frac{dN}{dx} = 0 \rightarrow N = \text{constant}$$

$$\Sigma F_y = 0 \rightarrow -V(x) + q\,dx + V(x) + \frac{dV}{dx}dx = 0 \rightarrow \frac{dV}{dx} = -q$$

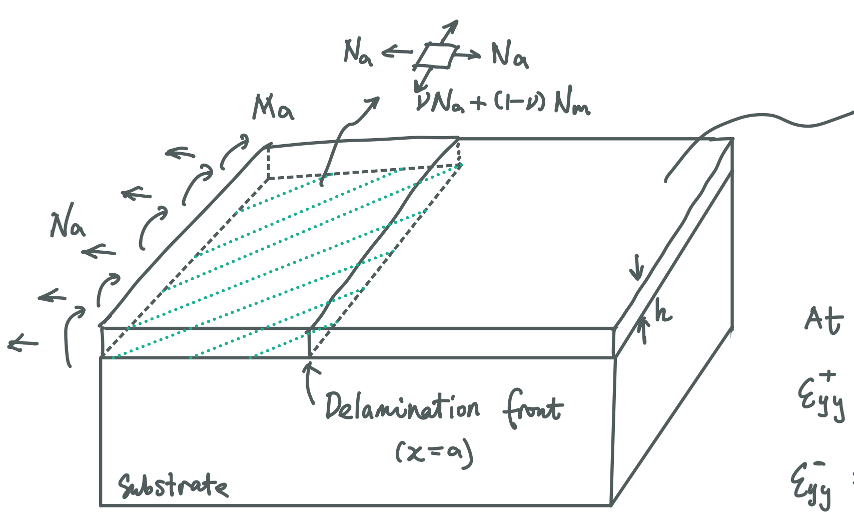
$$\Sigma M_z^{x+dx} = 0 \rightarrow -\cancel{M(x)} + V\cdot dx - N\cdot\left(\frac{dv}{dx}dx + \cancel{O(dx^3)}\right) - q\cdot\cancel{O(dx^2)} + \cancel{M(x)} + \frac{dM}{dx}dx + \cancel{HOT} = 0$$

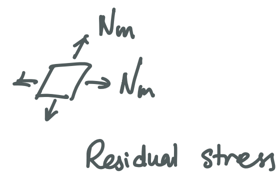
$$\rightarrow \frac{dM}{dx} - N\frac{dv}{dx} + V = 0 \rightarrow \frac{d^2M}{dx^2} - N\frac{d^2v}{dx^2} - q = 0$$

Finally, linear material law gives $M = B\kappa$, $\kappa = \frac{v''}{(1+v'^2)^{3/2}} \simeq v''$ for moderate rotations.

$$\therefore \boxed{\frac{d^4v}{dx^4} - \frac{N}{B}\frac{d^2v}{dx^2} = \frac{q}{B}}$$, where $N = E'h\varepsilon_{xx}$, $B = \frac{1}{12}E'h^3$, $E' = \frac{E}{1-\nu^2}$ in general.

We are interested in the energy release rate in this system. Consider a region that is close to the edge of the delamination zone. At this level of observation, the edge is essentially straight and the state of deformation is "generalized" plane strain.

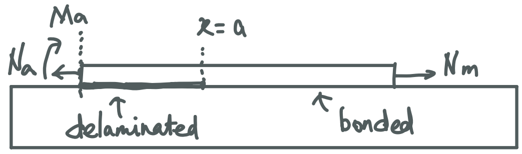




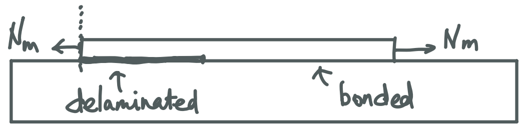
At $x = a$,

$$\varepsilon_{yy}^{+} = \frac{1-\nu}{Eh}N_m$$

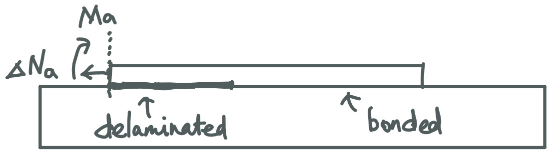
$$\varepsilon_{yy}^{-} = \frac{\cancel{\nu N_a} + (1-\nu)N_m - \cancel{\nu N_a}}{Eh} = \varepsilon_{yy}^{+}$$



The state (A) is the superposition of states (B) and (C).



- The driving force for delamination in (B) is zero.
- $\Pi = U_{SE} - W_\sigma$.



$\Delta N_a = N_a - N_m$

$$a \to a + \delta a, \quad \begin{cases} \delta U_{SE} = \frac{1}{2} \frac{\Delta N_a^2}{E'h} \delta a + \frac{1}{2} \frac{M_a^2}{B} \delta a \\ \delta W_\sigma = 2\, \delta U_{SE} \end{cases}$$

$$\to \mathcal{G} = -\frac{\delta \Pi}{\delta a} = \delta U_{SE} / \delta a$$

$$\therefore \boxed{\mathcal{G}(a) = \frac{1-\nu^2}{2Eh} \Delta N_a^2 + \frac{1}{2B} M_a^2}$$

"Local condition"

The energy release rate for advance of the delamination front is determined by the edge loads, ΔN_a and M_a, which are <u>not known</u> a priori in general (Need to solve the BVP).

According to Hutchison & Suo (1991), the stress intensity factors are

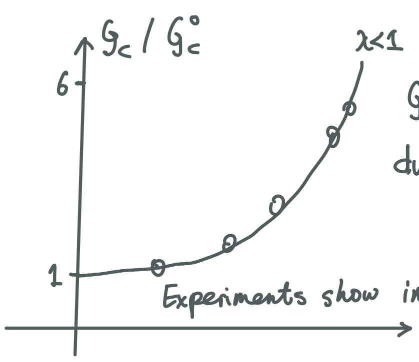
$$K_I = \frac{1}{\sqrt{2}} \left[\Delta N_a h^{-1/2} \cos\omega + 2\sqrt{3}\, M_a h^{-\frac{3}{2}} \sin\omega \right]$$

$$K_{II} = \frac{1}{\sqrt{2}} \left[\Delta N_a h^{-1/2} \sin\omega - 2\sqrt{3}\, M_a h^{-\frac{3}{2}} \cos\omega \right]$$

$$\to \underbrace{\tan\psi}_{\text{Mode-Mixity}} = \frac{Im\,(K h^{i\varepsilon})}{Re\,(K h^{i\varepsilon})} = \frac{\sqrt{12}\, M_a + \Delta N_a h \tan\omega}{\sqrt{12}\, M_a \tan\omega + \Delta N_a h}$$

$\omega = \omega(\underbrace{h/H}_{\text{substrate thickness}}, \varepsilon) \to 45° - 65°$ as $h/H \to 0$

where $K = K_I + iK_{II}$, $\varepsilon = \varepsilon(E, \nu, \underbrace{E_s, \nu_s}_{\text{substrate properties}})$

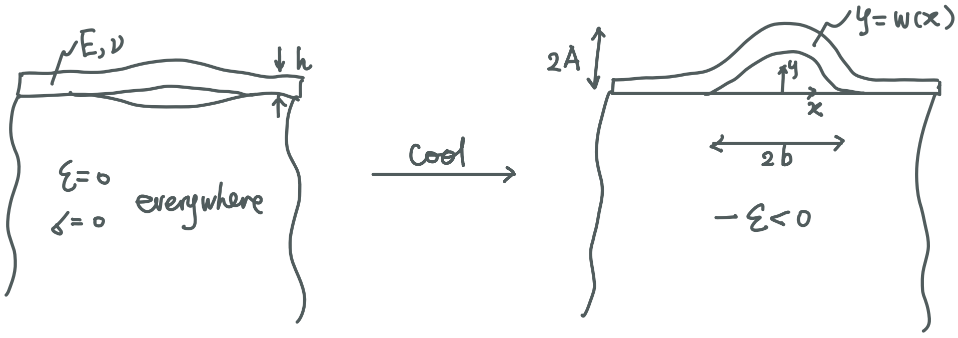


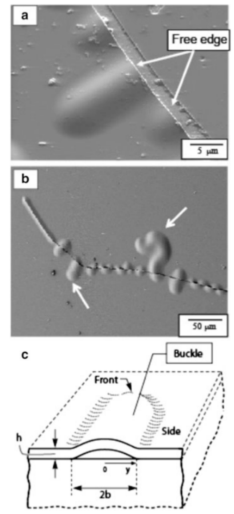
$$\mathcal{G}_c(\psi) = \mathcal{G}_c^1 \left[1 + (\lambda - 1) \sin^2\psi \right]^{-1}$$

due to asperity contact and plasticity.

Experiments show increased $\mathcal{G}_c$ as K_{II} increases.

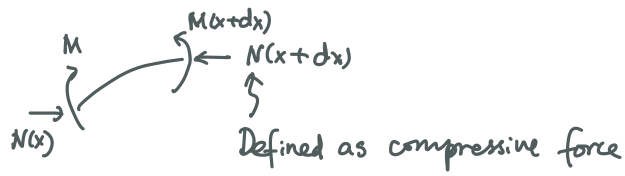
Buckle delamination





When to occur and what determines A & b?

Let us solve for this boundary value problem :



$$B\frac{d^4w}{dx^4} + N\frac{d^2w}{dx^2} = 0 \quad \& \quad N = \text{constant}$$

$$\rightarrow w = A + \underbrace{\cancel{C}}_{0}x + \underbrace{\cancel{D}}_{0}\sin\sqrt{\frac{N}{D}}\,x + F\cos\sqrt{\frac{N}{B}}\,x.$$

symmetry

Boundary conditions :

$$w(\pm b) = 0 \rightarrow A + F\cos\sqrt{\frac{N}{B}}\,b = 0 \longrightarrow F = A$$

$$w'(\pm b) = 0 \rightarrow \sin\sqrt{\frac{N}{B}}\,b = 0 \rightarrow N = \frac{\pi^2 B}{b^2}$$

(Recall the Euler instabity $P_{cr} = \frac{\pi^2 EI}{(\mu l)^2}$)

To determine A, we need to describe the axial strain of the centerline ε_{xx}

$$\varepsilon_{xx} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 = \frac{N}{E'h}$$

$$\underbrace{u(x=b) - u(x=-b)}_{-2b\cdot\varepsilon \text{ due to cooling}} = \int_{-b}^{b}\frac{du}{dx}dx = \underbrace{\int_{-b}^{b}\left[\frac{N}{E'h} - \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]dx}$$

$$= \underbrace{\frac{N}{E'h}}_{\varepsilon_c = \frac{\pi^2}{12}\left(\frac{h}{b}\right)^2}\cdot 2b - \underbrace{\int_{-b}^{b}\frac{N}{B}A^2\sin^2\sqrt{\frac{N}{B}}\,x\,dx}_{\frac{\pi^2}{2b^2}A^2\int_{-b}^{b}\sin^2\left(\frac{\pi}{b}x\right)dx = \frac{\pi^2}{2b}A^2}$$

$$\rightarrow -2b\varepsilon = 2b\varepsilon_c - \frac{\pi^2}{2b}A^2 \rightarrow A^2 = \frac{4b^2}{\pi^2}(\varepsilon - \varepsilon_c)$$

critical strain for buckling.

$$\therefore \boxed{w(x) = A\left(1 + \cos\frac{\pi}{b}x\right), \quad A = \frac{2b}{\pi}(\varepsilon - \varepsilon_c)^{1/2}, \quad \varepsilon_c = \frac{\pi^2}{12}\left(\frac{h}{b}\right)^2}$$

• Now, we know the solution for buckled film. Let's compute the energy release rate.

$$U_{flat} = \frac{1}{2}E'\varepsilon^2 h(\ell-2b)$$, where ℓ is the total length of the film

$$U_{buckle} = \int_{-b}^{b} \frac{1}{2}B(w'')^2 + \frac{1}{2}E'h(\varepsilon_{xx})^2\,dx$$

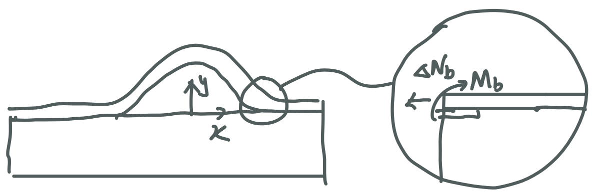
$$= \frac{1}{2}\cdot\frac{1}{12}E'h^3\cdot\frac{\pi^4}{b^4}\cdot A^2\underbrace{\int_{-b}^{b}\cos^2\frac{\pi}{b}x\,dx}_{b} + \frac{1}{2}E'h\,\frac{\overbrace{N^2}^{\frac{\pi^4}{b^4}\cdot\frac{1}{144}E'^2h^6}}{(E'h)^2}\cdot 2b$$

$$= \underbrace{\frac{1}{24}E'h^3\frac{\pi^4}{b^3}\cdot\frac{4b^2}{\pi^2}}_{\frac{\pi^2}{6}E'h\cdot\frac{h^2}{b}}(\varepsilon-\varepsilon_c) + \frac{\pi^4}{144}\frac{E'h}{b^3}\cdot h^4$$

$$\rightarrow U_{SE} = \frac{E'h}{2}\left[(\ell-2b)\varepsilon^2 + \frac{\pi^2}{3}\frac{h^2}{b}\varepsilon\underbrace{-\frac{\pi^2}{3}\frac{h^2}{b}\cdot\frac{\pi^2}{12}\left(\frac{h}{b}\right)^2 + \frac{\pi^4}{72}\frac{h^4}{b^3}}_{-\frac{\pi^4}{72}\frac{h^4}{b^3}}\right]$$

$$\mathcal{G} = -\frac{\partial U_{SE}}{\partial(2b)} = \frac{E'h}{2}\left(\varepsilon^2 + \underbrace{\frac{\pi^2}{6}\frac{h^2}{b^2}\varepsilon}_{2\varepsilon\varepsilon_c} - \underbrace{\frac{\pi^4}{48}\frac{h^4}{b^4}}_{3\varepsilon_c^2}\right) = \frac{1}{2}E'h(\varepsilon+3\varepsilon_c)(\varepsilon-\varepsilon_c)$$

• We can also obtain this according to the local observation

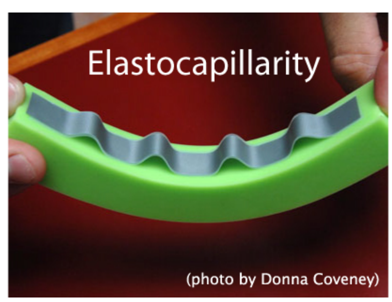


$$\Delta N_b = (-N) - (-E'h\varepsilon) = -\frac{\pi^2 B}{b^2} + E'h\varepsilon\ ; \qquad M_b = Bw''(x=b) = +BA\frac{\pi^2}{b^2}$$

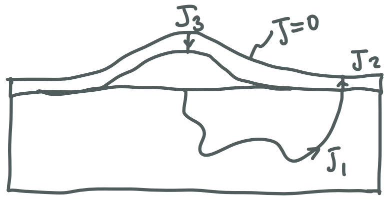
$$\text{P90} \rightarrow \mathcal{G} = \underbrace{\frac{1}{2E'h}\Delta N_b^2}_{\mathcal{G}_s} + \underbrace{\frac{1}{2B}M_b^2}_{\mathcal{G}_b} = \frac{E'h}{2}\left(\varepsilon^2 + \frac{\pi^2}{6}\frac{h^2}{b^2}\varepsilon - \frac{\pi^4}{48}\frac{h^4}{b^4}\right) \checkmark = \frac{\pi^4}{96}\frac{E'h(3A^4+4A^2h^2)}{b^4}\ ?$$

As the substrate is very soft (Pan et al IJSS, 2014), or the interface is slippery (Dai et al. JMPS, 2020) so that $\Delta N_b \rightarrow 0$, $\mathcal{G}_s$ is not important:

$$\Gamma = \frac{1}{2B}M_b^2 = \frac{\pi^4}{2}\frac{BA^2}{b^4} \xrightarrow[2b=\lambda]{2A=\delta} 2\pi^4\frac{B\delta^2}{\lambda^4}$$ (D. Vella et al. PNAS 2009)



- Lastly, let's try J integral.



$$J = \int_\Gamma W n_1 - \sigma_{ij} n_j u_{i,1}\, d\Gamma$$

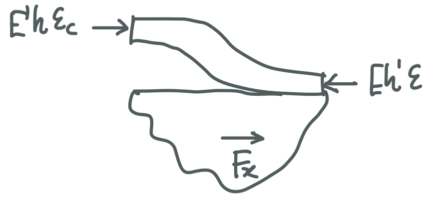
$$J_2 = \int_0^h W - \sigma_{xx}\varepsilon_{xx}\, dy = -\frac{1}{2}E'h\varepsilon^2$$

$$J_3 = \int_{-h}^{0} -\overset{n_1=-1}{W} + \sigma_{xx}\varepsilon_{xx}\, d(-y)$$

$$= \int_0^h \frac{1}{2}\sigma_{xx}\varepsilon_{xx}\, dy = \int_0^h \frac{1}{2}\left(\overset{\text{stretching}}{\sigma_{xx}^s} + \overset{\text{bending}}{\sigma_{xx}^b}\right)\left(\varepsilon_{xx}^s + \varepsilon_{xx}^b\right) dy$$

$$= \frac{1}{2}B(w'')^2 + \frac{1}{2}E'h\varepsilon_{xx}^2 \leftarrow = \frac{N}{E'h} = \varepsilon_c$$

$$= \frac{1}{24}E'h^3 \underbrace{\frac{4b^2}{\pi^2}(\varepsilon-\varepsilon_c)}_{A^2}\cdot\frac{\pi^4}{b^4} + \frac{1}{2}E'h\varepsilon_c^2 = \frac{1}{2}E'h\left(4\varepsilon\varepsilon_c - 3\varepsilon_c^2\right)$$



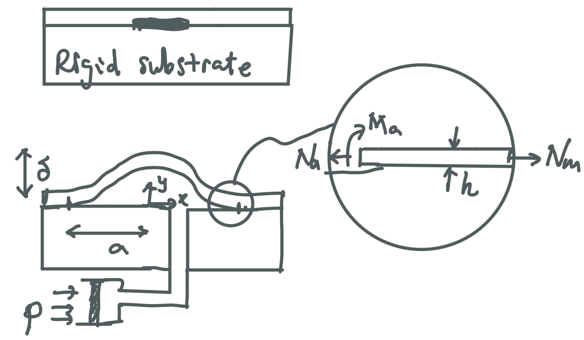
$$J_1 = \int_{\Gamma_1} \cancelto{0,\text{ rigid}}{W n_1} - t_x \cancelto{-\varepsilon}{\varepsilon_{xx}} - t_y \cancelto{0 \text{ rigid}}{\varepsilon_{xy}}\, d\Gamma$$

$$= \varepsilon\int_\Gamma t_x\, d\Gamma = \varepsilon F_x = Eh'\left(\varepsilon^2 - \varepsilon\varepsilon_c\right)$$

$$\rightarrow \mathcal{G} = J = J_1 + J_2 + J_3 = \frac{Eh'}{2}\left(-\varepsilon^2 + 4\varepsilon\varepsilon_c - 3\varepsilon_c^2 + 2\varepsilon^2 - 2\varepsilon\varepsilon_c\right) = \frac{E'h}{2}\left(\varepsilon^2 + 2\varepsilon\varepsilon_c - 3\varepsilon_c^2\right) \checkmark$$

Pressurized bulge of uniform width

The straight-sided bulge configuration is perhaps of less practical significance than the circular case. But the mechanical response of the film can be described in a fairly "transparent" way at various levels of approximations — useful for introducing ideas.



Now the deflection results from external loading p (positively defined upward)

$$B\frac{d^4w}{dx^4} - N\frac{d^2w}{dx^2} = p$$

There are three sources of elastic energy: bending, stretching & residual stress. Let us consider a scaling argument.

Geometry: $K \sim \delta/a^2$, $\varepsilon \sim \delta^2/a^2$

Bending energy: $U_b \sim BK^2 \sim B\delta^2/a^4$ (per area)

Stretching energy: $U_s \sim N\varepsilon \sim (\bar{E}h\varepsilon + N_m)\varepsilon \sim \begin{cases} \bar{E}h\delta^4/a^4 & \text{as } \frac{N_m}{\bar{E}h} \ll \frac{\delta^2}{a^2} \quad \text{(Membrane)} \\ N_m\delta^2/a^2 & \text{as } \frac{N_m}{\bar{E}h} \gg \frac{\delta^2}{a^2} \quad \text{(Pretension)} \end{cases}$

• The bending response ($U_b \gg U_s$)

This case leads to the simplest level of approximation — linear plate theory

$$B\frac{d^4w}{dx^4} = p$$

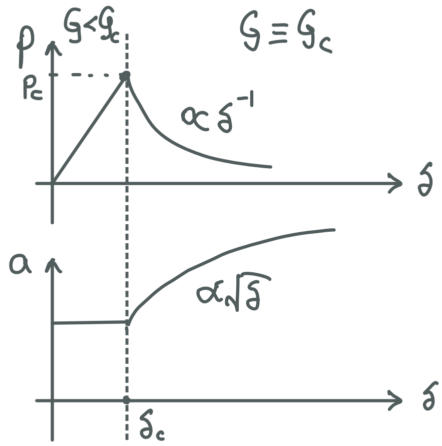
$w(x=\pm a) = 0$ $\Rightarrow$ $w(x) = \dfrac{pa^4}{24B}\left(1 - \dfrac{x^2}{a^2}\right)^2$ & $p = \dfrac{24B\delta}{a^4}$ (linear p-δ relation)

$w'(x=\pm a) = 0$

Accurate when $B\delta^2/a^4 \gg \{\bar{E}h\delta^4/a^4,\ N_m\delta^2/a^2\}$, i.e., $\delta \ll \left(\frac{B}{\bar{E}h}\right)^{1/2} \sim h$ and $\frac{N_m}{\bar{E}h} \ll \frac{B}{\bar{E}ha^2} \sim \frac{h^2}{a^2}$

This "configurational" driving force for delamination at the edge of the pressurized zone can be calculated by Eq on P106 with $\Delta N_a = 0$

$$\mathcal{G} = \underbrace{\frac{1}{2E'h}\Delta N_a^2}_{\sim E'h\frac{\delta^4}{a^4}} + \underbrace{\frac{1}{2}B(w'')_a^2}_{\ll \sim E'h^3\frac{\delta^2}{a^4}} = \frac{1}{18}\frac{p^2a^4}{B} = \frac{32B\delta^2}{a^4}$$



• "Large deflection" response ($U_b \sim U_s$)

If the center point deflection δ increases to values on the order of h, we need to consider the generated membrane stress in the film due to transverse deflection (in addition to residual membrane stress). Here we consider a simplified case in which $N_m = 0$.

$$B\frac{d^4w}{dx^4} - N\frac{d^2w}{dx^2} = p \longrightarrow w(x) = -\frac{p}{2N}x^2 + A + \cancel{B}x + \cancel{C}\sinh\sqrt{\tfrac{N}{B}}x + D\cosh\sqrt{\tfrac{N}{B}}x$$

due to symmetry 0 0

Boundary conditions : $w(x=\pm a) = w'(x=\pm a) = 0$

$$\left.\begin{aligned} -\frac{p}{2N}a^2 + A + D\cosh\sqrt{\tfrac{N}{B}}a &= 0 \\ -\frac{p}{N}a + D\sqrt{\tfrac{N}{B}}\sinh\sqrt{\tfrac{N}{B}}a &= 0 \end{aligned}\right\} \xrightarrow{\tau = \left(\frac{Na^2}{B}\right)^{1/2}} \begin{aligned} A &= \frac{pa^4}{2B}\,\frac{\tau - 2\coth\tau}{\tau^3} \\ D &= \frac{pa^4}{B}\,\frac{1}{\tau\sinh\tau} \end{aligned}$$

We obtain $w(x) = \dfrac{pa^4}{B}\left[\dfrac{\tau - 2\coth\tau}{2\tau^3} - \dfrac{1}{\tau^2}\left(\dfrac{x}{a}\right)^2 + \dfrac{\cosh\left(\frac{\tau x}{a}\right)}{\tau^3\sinh\tau}\right]$

$$\longrightarrow \delta = \frac{pa^4}{B}\left(\frac{\tau - 2\coth\tau}{2\tau^3} + \frac{1}{\tau^3\sinh\tau}\right) = \begin{cases} \dfrac{pa^4}{24B}\left[1 - \dfrac{1}{10}\tau^2 + O(\tau^4)\right], & \text{for } \tau \ll 1 \text{ (Plate response)} \\ \dfrac{pa^2}{2N}\left[1 - \dfrac{2\coth\tau}{\tau} + O(e^{-\tau}\tau^{-3})\right], & \text{for } \tau \gg 1 \text{ (Membrane response)} \end{cases}$$

What is membrane response as $\Gamma \gg 1$? Imagine zero-bending modulus plate: $-N\frac{d^2w}{dx^2}=P$, its solution is simply $w=\frac{Pa^2}{2N}\left(1-\frac{x^2}{a^2}\right)$. It satisfies $w(\pm a)=0$ but not $w'(\pm a)=0$!

Note that we still don't know what N or Γ is! Need to use BCs about in-plane displacement.

$$\frac{N}{E'h}=\varepsilon_{xx}=\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2 \to \frac{du}{dx}=\frac{N}{E'h}-\frac{1}{2}\left(\frac{dw}{dx}\right)^2$$

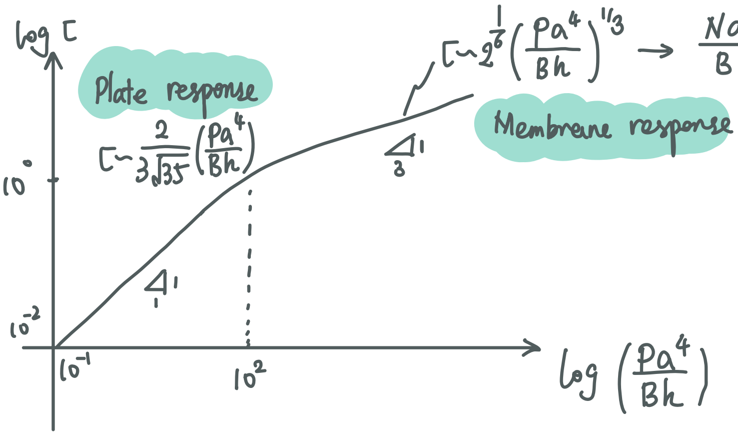
since $N_m=0$

$$\to u(a)-u(-a)=\cancel{\varepsilon_m' 2a}^{\,0}=\int_{-a}^{a}\left[\frac{B\Gamma^2}{E'ha^2}-\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]dx$$

$$=\frac{h^2}{6a}\Gamma^2-\frac{P^2a^7}{6B^2}\,\frac{12+2\Gamma^2-9\Gamma\coth\Gamma-3\Gamma^2/\sinh^2\Gamma}{\Gamma^6}$$

$$\to \Gamma^8=\left(\frac{Pa^4}{Bh}\right)^2\left(12+2\Gamma^2-9\Gamma\coth\Gamma-3\Gamma^2/\sinh^2\Gamma\right) \quad \text{or} \quad \frac{Pa^4}{Bh}=f(\Gamma)$$

$$\Gamma\sim 2^{\frac{1}{6}}\left(\frac{Pa^4}{Bh}\right)^{1/3} \to \frac{Na^2}{B}\sim 2^{1/3}\left(\frac{Pa^4}{Bh}\right)^{2/3} \to N\sim\frac{P^{2/3}a^{2/3}(E'h)^{1/3}\cancel{h^{2/3}}}{\cancel{h^{2/3}}\;6^{1/3}}$$



- When $\frac{Pa^4}{Bh}\sim\frac{\delta}{h}\ll 1$, $\Gamma\to 0$, $\delta=\frac{Pa^4}{24B}$ or $P=\frac{24B}{a^4}\delta$ (Plate)
- When $\frac{Pa^4}{Bh}\sim\frac{\delta}{h}\gg 1$, $\delta=\frac{Pa^2}{2N}=\frac{6^{1/3}}{2}\frac{P^{1/3}a^{\frac{4}{3}}}{(E'h)^{1/3}}=\left(\frac{3Pa^4}{4E'h}\right)^{1/3}$ or $P=\frac{4}{3}\frac{E'h}{a^4}\delta^3$ (Membrane)

Now we are able to determine ΔN_a and M_a in terms of Γ, specifically

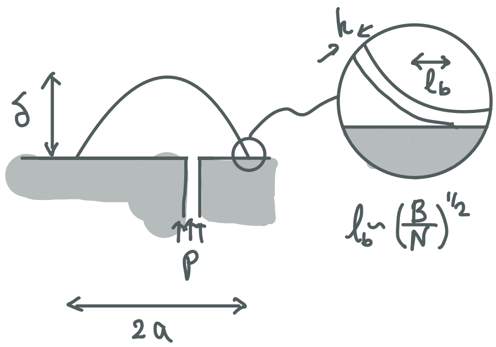
$$\Delta N_a=N=\frac{B\Gamma^2}{a^2},\quad M_a=Bw''(x=a)=\frac{Pa^2}{\Gamma^2}(1-\Gamma\coth\Gamma)=\frac{Bh}{a^2}\underbrace{f(\Gamma)\frac{1-\Gamma\coth\Gamma}{\Gamma^2}}_{=g(\Gamma)}$$

$$G=\frac{1}{2}\frac{\Delta N_a^2}{E'h}+\frac{1}{2}\frac{M_a^2}{B}=\frac{B^2}{2E'ha^4}\Gamma^4+\frac{Bh^2}{2a^4}g^2(\Gamma)=\frac{E'h^5}{a^4}\frac{\Gamma^4}{288}\left[1+\frac{12(1-\Gamma\coth\Gamma)^2}{2(6+\Gamma^2)-9\Gamma\coth\Gamma-3\Gamma^2\operatorname{csch}^2\Gamma}\right]$$

$$\sim \begin{cases} \dfrac{E'h^5}{a^4}\cdot\dfrac{35}{96}\tau^2 = \dfrac{Bh^2}{a^4}\,\dfrac{35}{8}\cdot\dfrac{4}{9\cdot 35}\,\dfrac{p^2a^8}{B^2h^2} = \dfrac{p^2a^4}{18B}, \text{ as } \tau \ll 1 & \text{(Plate limit)} \checkmark \\[2ex] \dfrac{E'h^5}{a^4}\cdot\dfrac{7}{288}\tau^4 = \dfrac{7}{2\times 6^{2/3}}\left(\dfrac{p^4 a^4}{E'h}\right)^{1/3}, \text{ as } \tau \gg 1 & \text{(Membrane limit? Need to check)} \end{cases}$$

• Membrane response ($U_s \gg U_b$)

Still consider $N_m = 0$ so that $U_s \gg U_b$ means $E'h\dfrac{\delta^4}{a^4} \gg B\dfrac{\delta^2}{a^4}$, i.e., $\delta \gg \sqrt{\dfrac{B}{E'h}} \sim h$



Need to be careful about l_b – two ways to analyze:

Prescribed p or prescribed δ. Let's do the latter.

$$\varepsilon_{xx} \sim \frac{\delta^2}{a^2} \rightarrow N \sim E'h\frac{\delta^2}{a^2} \rightarrow l_b \sim \left(\frac{B}{E'h\delta^2/a^2}\right)^{1/2} \sim \frac{h}{\delta}a \ll a$$

Non-dimensionalization

$$X = \frac{x}{a}, \quad W = \frac{w}{\delta}, \quad \tilde{N} = \frac{N}{E'h\delta^2/a^2}, \quad P = \frac{p}{E'h\delta^2/a^2 \times \delta/a^2} = \frac{pa^4}{E'h\delta^3}$$

$$B\frac{d^4w}{dx^4} - N\frac{d^2w}{dx^2} = p \longrightarrow \frac{B\cdot\delta}{a^4}W_{XXXX} - \tilde{N}\frac{E'h\delta^2}{a^2}\cdot\frac{\delta}{a^2}W_{XX} = P\cdot\frac{E'h\delta^3}{a^4}$$

$$\therefore \ \epsilon^2 W_{XXXX} - \tilde{N}W_{XX} = P, \text{ where } \epsilon = \left(\frac{B}{E'h\delta^2}\right)^{1/2} \sim \frac{h}{\delta} \ll 1$$

This definition gives $l_b \sim \epsilon a$

Since $\epsilon^2 \lll 1$, we neglect the high order term and immediately have the solution:

$$W = \frac{P}{2\tilde{N}}(1-X^2) \quad \text{or} \quad w(x) = \underbrace{\frac{pa^2}{2N}}_{=\delta \text{ since } w(0)=\delta}\left(1-\frac{x^2}{a^2}\right) \text{ in dimensional form}$$

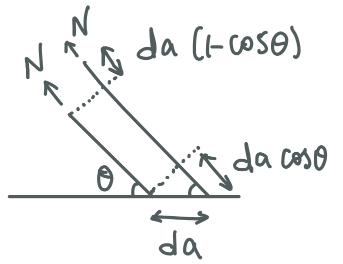
where we have used $W(\pm 1) = 0$. To calulate $\tilde{N}$, recall that

$$\frac{N}{E'h} = \varepsilon_{xx} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \rightarrow u(a) - u(-a) = 0 = \int_{-a}^{a}\left[\frac{N}{E'h} - \frac{1}{2}\left(2\frac{\delta x}{a^2}\right)^2\right]dx$$

$\rightarrow N = \frac{2}{3}E'h\frac{\delta^2}{a^2}$ and $p = \frac{2N\delta}{a^2} = \frac{4}{3}E'h\frac{\delta^3}{a^4}$ (Agree with results on Page 96)

What is the energy release rate? Two perspectives!

① From the point of view of $-\frac{\partial\Pi}{\partial a}$. (Kendall's peeling angle)



(Local view of membrane)

$$d\Pi = dU_{SE} - dW_s$$

$$= da \times \frac{1}{2}\frac{N^2}{E'h} - N\cdot da(1-\cos\theta) - N\cdot \overset{\text{Stretching}}{\overbrace{\frac{N}{E'h}}}da$$

$$\rightarrow \boxed{G = -\frac{d\Pi}{da} = N(1-\cos\theta) + \frac{1}{2}\frac{N^2}{E'h}} \quad ⊛$$

$$= \frac{1}{2}w'^2\Big|_a N + \frac{1}{2}\frac{N^2}{E'h}$$

$$= \frac{1}{2}\cdot\left(\frac{2\delta}{a}\right)^2\cdot\frac{2}{3}E'h\frac{\delta^2}{a^2} + \frac{1}{2}E'h\left(\frac{2}{3}\frac{\delta^2}{a^2}\right)^2$$

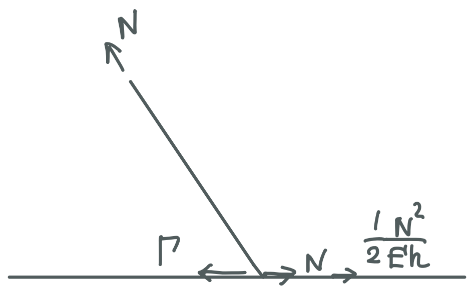
$$= \frac{14}{9}E'h\frac{\delta^4}{a^4}$$

$$= \frac{14}{9}E'h\frac{1}{a^4}\left(\frac{3pa^4}{4E'h}\right)^{\frac{4}{3}}$$

$$= \frac{7}{2\times 6^{2/3}}\left(\frac{p^4 a^4}{E'h}\right)^{1/3}$$

(Agree with results on Page 97)

- θ is constant if delamination occurs
- Contribution by p is neglected since the level of observation is small



Geometric interpretation of ⊛

It should be noted here that ⊛ applies to the peeling problem with ang θ.

In particular, when θ is not too close to $0°$, $N \gg \frac{N^2}{E'h}$, we have $G = N(1-\cos\theta)$, i.e., peeling at $\theta = \frac{\pi}{2}$, $G_c = P_c$.

This result is neat and nice, but it does not give anything at K_I, K_{II} or ψ, which needs information at ΔN_a and M_a.

② From the point of view of boundary layer analysis

Want to understand what is going on near $x = \pm a$. Return the unsimplified equation at the level of observation $\sim l_b$, i.e., ϵ in dimensionless form.

$$\xi = \frac{x-1}{\epsilon}, \quad \frac{d}{dx} = \frac{d}{d\xi}\cdot\frac{1}{\epsilon} \rightarrow \frac{\epsilon^2}{\epsilon^4} W_{\xi\xi\xi\xi} - \frac{\tilde{N}}{\epsilon^2} W_{\xi\xi} = P \rightarrow W_{\xi\xi\xi\xi} - \tilde{N} W_{\xi\xi} = \cancel{P\epsilon^2}^{\,0}$$

(pressure not important here)

The solution is

$$W(\xi) = C_1 + C_2\xi + C_3 e^{+\tilde{N}^{1/2}\xi} + \cancel{C_4}^{\,0} e^{-\tilde{N}^{1/2}\xi}$$

Boundary and matching conditions:

$$W_x(x\rightarrow 1) = \frac{-P}{\tilde{N}} = W_\xi \cdot \frac{1}{\epsilon}$$

- At $x \rightarrow a$, $\xi \rightarrow 0$, $W = W' = 0$
- At $x \rightarrow a^-$, $\xi \rightarrow -\infty$, W' finite & $W' \rightarrow -\frac{\epsilon P}{\tilde{N}} = C_2$

$$\rightarrow W(\xi) = -\frac{P\epsilon}{\tilde{N}}\xi + \frac{P\epsilon}{\tilde{N}^{3/2}}\left(e^{\tilde{N}^{1/2}\xi} - 1\right)$$

$$W''(\xi) = \frac{P\epsilon}{\tilde{N}^{1/2}} e^{\tilde{N}^{1/2}\xi} = \frac{4/3}{\sqrt{2/3}}\epsilon\, \mathrm{Exp}\left[\frac{\sqrt{2}}{\sqrt{3}}\frac{(x-1)}{\epsilon}\right]$$

(By a few $\epsilon = \frac{h}{\delta}$ away from the edge, the curvature decays to zero)

$w'' \times \frac{a^2}{\delta} \times \epsilon^2$

from leading order solution

Therefore, $\Delta N_a = \frac{2}{3} E'h \frac{\delta^2}{a^2}$.

$$M_a = B w''(x=a) = B\cdot\frac{\delta}{a^2}\frac{1}{\epsilon^2}\cdot W''(\xi=0) = B\frac{\delta}{a^2}\left(\frac{E'h\delta^2}{B}\right)^{1/2}\frac{4}{6^{1/2}} = \left(\frac{8}{3}\frac{BE'h\delta^4}{a^4}\right)^{1/2}$$

$$\rightarrow \mathcal{G} = \frac{1}{2}\frac{\Delta N_a^2}{E'h} + \frac{1}{2}\frac{M_a^2}{B} = \frac{2}{9}E'h\frac{\delta^4}{a^4} + \frac{8}{2\times 3}E'h\frac{\delta^4}{a^2} = \frac{14}{9}E'h\frac{\delta^4}{a^2} \quad ✓$$

$$\tan\psi = \frac{\sqrt{12}M_a + \Delta N_a h \tan\omega}{\sqrt{12}M_a\tan\omega + \Delta N_a h} = \frac{\sqrt{6} + \tan\omega}{1 - \sqrt{6}\tan\omega} \sim -1.2 - -0.8 \text{ for } 45^\circ < \omega < 65^\circ$$

Jensen (1993)