Problem 1: Asymptotic stress field for anti-plane shear problem. Derive the form of the singular near-tip stress field for mode III conditions in an isotropic linear elastic solid. After you show that $p = -1/2$, the undetermined constant should be redefined such that the shear stress on $\theta = 0$ is written as $\sigma_{23}(r, \theta = 0) = K_{\mathrm{III}}/\sqrt{2\pi r}$. You will need to derive the governing equation for an anti-plane shear problem, solve this equation using the separation of variables, and apply the appropriate boundary conditions. Do any $T-$stress terms exist corresponding to $p = 0$? Verify that your results agree with those from a trusted reference. Perform a crack closure integral to determine the relationship between $\mathcal{G}$ and K_{III} for pure anti-plane shear loading.

Problem 2: Stress intensity factors for a penny-shaped crack. From the consideration of an ellipsoidal void, Eshelby showed that the discontinuous part of the displacements on the surface of a penny-shaped crack given by the surface $x^2 + y^2 = r^2 < a^2$ in a linear elastic isotropic body are

$$u_x = \pm\alpha\frac{\tau}{\mu}\sqrt{a^2 - r^2} \quad \text{and} \quad u_z = \pm\beta\frac{\sigma}{\mu}\sqrt{a^2 - r^2}$$

where the loading in the far field is

$$\sigma_{zz} = \sigma \quad \text{and} \quad \sigma_{zx} = \tau.$$

Use these results to show that

$$\begin{aligned} K_{\mathrm{I}} &= \frac{2}{\pi}\sigma\sqrt{\pi a}, \\ K_{\mathrm{II}} &= \frac{4}{\pi(2-\nu)}\tau\sqrt{\pi a}\cos\theta, \\ K_{\mathrm{III}} &= \frac{4(1-\nu)}{\pi(2-\nu)}\tau\sqrt{\pi a}\sin\theta, \end{aligned}$$

around the tip of a penny-shaped crack. Note that θ is the angle between the x-axis and the radial direction in the $x - y$ plane. In order to determine α and β you will need to use the fact that the change in potential energy of the system due to the penny-shaped crack is

$$\Delta PE = -\frac{\sigma}{2}\int_0^{2\pi}\left[\int_o^a \left(u_z^+ - u_z^-\right) r\,\mathrm{d}r\right]\mathrm{d}\theta - \frac{\tau}{2}\int_0^{2\pi}\left[\int_o^a \left(u_x^+ - u_x^-\right) r\,\mathrm{d}r\right]\mathrm{d}\theta.$$

Have you come other methods from elasticity theory to determine α and β? (Don't have to answer.)