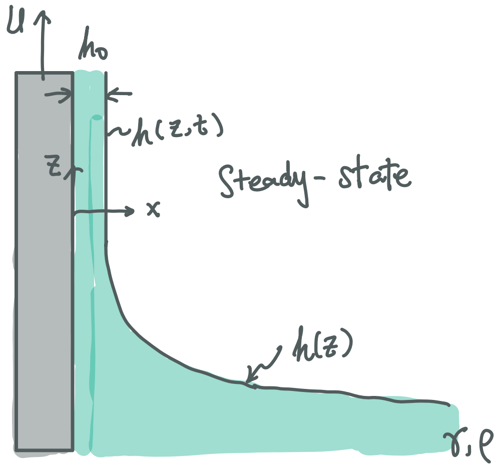
The Landau–Levich problem (1942)



Drawing a sheet at a constant speed out of a bath of liquid.

- Review modelling of thin films
- Introduce method of matched asymptotic expansion

• The dynamic equation

From previous lecture, we know:

$$\left.\begin{aligned} &F(x,z,t) = x - h(z,t) = 0 \\ &\nabla \cdot \underset{\sim}{u} = 0 \end{aligned}\right\} \longrightarrow \underbrace{\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial z} = 0}_{\text{vertical drawing}}, \quad Q = \int_0^h \underbrace{u}_{\text{vertical velocity}} \, dx$$

$$\rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u}\right) = -\nabla p + \mu\nabla^2\underset{\sim}{u} + \rho\underset{\sim}{g} \xrightarrow[Re \ll 1]{} \frac{\partial p}{\partial x} = 0 \rightarrow p = p(z)$$

$$[z] \gg [x]$$

$$-\frac{\partial p}{\partial z} + \mu\frac{\partial^2 u}{\partial x^2} + \cancel{\mu\frac{\partial^2 u}{\partial z^2}} - \rho g = 0$$

We now have a slightly different form of $\underset{\sim}{u}$ field:

$$u = +\frac{1}{2\mu}\left(\frac{\partial p}{\partial z} + \rho g\right)x^2 + C_1 x + C_2$$

On $x = 0$, $u(0) = C_2 = U$

On $x = h$, $\dfrac{\partial u}{\partial x} = \dfrac{2}{2\mu}\left(\dfrac{\partial p}{\partial z} + \rho g\right)h + C_1 = 0$

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial z} + \rho g\right)(x^2 - 2hx) + U$$

Use continuity condition to show that

$$Q = \int_0^h u \, dx = -\frac{1}{3\mu}\left(\frac{\partial p}{\partial z} + \rho g\right) h^3 + Uh$$

and, with $p = p_{atm} - \gamma \frac{\partial^2 h}{\partial z^2}$,

$$\frac{\partial h}{\partial t} + U h_z + \frac{1}{3\mu}\frac{\partial}{\partial z}\left[h^3\left(\gamma \frac{\partial^3 h}{\partial z^3} - \rho g\right)\right] = 0$$

Natural to use U to rescale $\underset{\sim}{u}$, but there is no intrinsic length scale (h_0 is nice but it is unknown a priori). So use an arbitrary ℓ first.

$$T = t/t_*, \quad H = h/\ell, \quad Z = z/\ell$$

$$\frac{\partial H}{\partial T} \cdot \frac{\ell}{t^*} + U H_Z + \frac{1}{3\mu \cancel{\ell} \partial Z}\left[\cancel{\ell^3} H^3 \left(\gamma H_{ZZZ} \cancel{\ell^{-2}} - \underbrace{\frac{\rho g \ell^2}{\gamma}}_{Bo} \cancel{\gamma \ell^{-2}}\right)\right] = 0$$

We rewrite as

$$\boxed{\frac{\partial H}{\partial T} + H_Z + \left[\frac{H^3}{3 Ca}\left(H_{ZZZ} - Bo\right)\right]_Z = 0}$$

by choosing $t^* = \ell / U$ and

$$Ca \equiv \frac{\mu U}{\gamma} \quad \text{(Capillary number)}$$

measuring the strength of viscous force $(\sim \mu U/\ell)$ compared to capillary forces $(\sim \gamma h_{xx} \sim \gamma/\ell)$.

• The steady state problem (Scaling wise)

h_0 – the film thickness at $z \to \infty$ is of most interest.

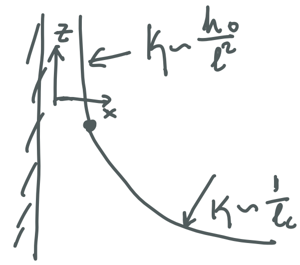
• Balancing gravity and viscosity

$$\frac{\rho g h_0}{\mu U / h_0} \sim 1 \to h_0 \sim \left(\frac{\mu U}{\rho g}\right)^{1/2} \sim \boxed{l_c \, Ca^{1/2}}$$

This can also be obtained by $\cancel{-\nabla p} + \mu \frac{\partial^2 u}{\partial x^2} + \rho \underset{\sim}{g} = 0$

• Balancing capillarity and viscosity

$$\nabla p \sim \frac{\partial}{\partial z}\left(\gamma \frac{\partial^2 h}{\partial z^2}\right) \sim \gamma \frac{h_0}{l^3}$$



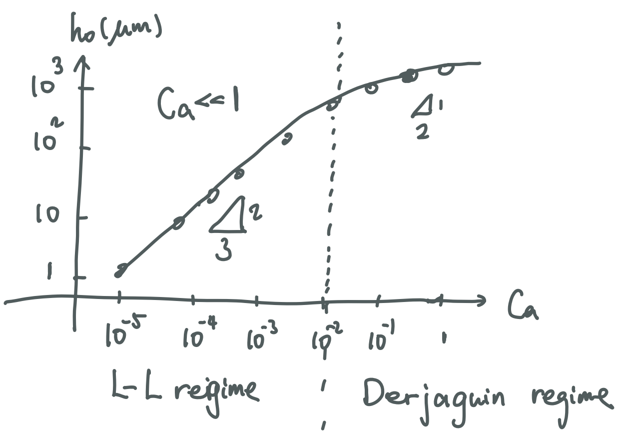
$$\mu \frac{\partial^2 u}{\partial x^2} \sim \mu \frac{U}{h_0^2}$$

Matching conditions between the film and meniscus gives

$$\frac{h_0}{l^2} \sim \frac{1}{l_c} \to l \sim (h_0 l_c)^{1/2}$$

$$\gamma h_0^{-\frac{1}{2}} l_c^{-\frac{3}{2}} \sim \mu U h_0^{-2} \to h_0 \sim \boxed{l_c \, Ca^{2/3}}$$

In experiments

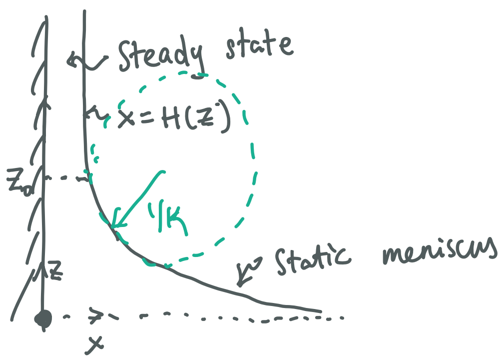


• The steady state problem for Ca<<1 (Landau-Levich)

Static meniscus

Haven dисcussed how $l_c\, Ca^{\alpha}$ relation, may choose $l = l_c$ to solve the problem.

$$Bo = \frac{\rho g l_c^2}{\gamma} = 1 \rightarrow \frac{\partial H}{\partial T} + H_z + \left[\frac{1}{3Ca} H^3 (H_{zzz} - 1)\right]_z = 0$$



In the meniscus regime, thin film approximations break down. Then use more general Young-Laplace equation.

$$P_{atm} + \gamma \nabla \cdot \underset{\sim}{n} + \rho g z = P_{atm}$$

The non-dimensionalized version reads

$$\frac{H_{zz}}{(1 + H_z^2)^{3/2}} - Z = 0$$

subject to

$$H(Z_0) = \text{Const.}$$

$$H'(Z_0) = 0$$

$$H \to \infty, \quad H_z \to -\infty \quad \text{as } Z \to 0$$

where Z_0 is the apparent contact point above the free surface.

Integrating once gives:

$$\frac{H_z}{(1 + H_z^2)^{1/2}} = \frac{1}{2} z^2 + C = \frac{1}{2}(z^2 - z_0^2)$$

after using $H'(z_0)=0$. Rearranging gives

$$\frac{H_z^2}{1+H_z^2} = 1-\frac{1}{1+H_z^2} = \frac{1}{4}(z^2-z_0^2)^2 \rightarrow H_z = \frac{z^2-z_0^2}{\left[4-(z_0^2-z^2)^2\right]^{1/2}} < 0$$

As $z \to 0$ we find

$$H_z = \frac{-z_0^2}{(4-z_0^4)^{1/2}} \rightarrow -\infty \quad \text{only if } z_0=\sqrt{2}$$

We must choos $z_0=\sqrt{2}$. This is now enough for us to deterime the local behavior near $z=z_0$ (which is what we need to match with the thin film region).

$$H_{zz}(z_0) = z_0 = \sqrt{2} \qquad \left(K \sim \frac{1}{\ell_c} \checkmark \right)$$

Locally, we have $H(z) = H(z_0) + H_z(z_0)(z-z_0) + \frac{1}{2}H_{zz}(z_0)(z-z_0)^2 + \cdots$, i.e.,

$$H(z) = \frac{(z-z_0)^2}{\sqrt{2}}, \quad \text{as } z \to z_0$$

The wall region

As $z \to z_0$, the fluid form a thin film on the wall. The steady state equation now is

$$H_z + \left[\frac{1}{3Ca} H^3 (H_{zzz}-1)\right]_z = 0.$$

Let's first see any simplification that can be made by $Ca \ll 1$. We are interested in the behavior as we approach z_0, so pose the rescaled vertical length near z_0:

not clear at this moment.

$$z = \underbrace{z_0 + \epsilon \bar{z}} + \epsilon^2 \hat{z} + \epsilon^3 \hat{\hat{z}} \cdots\cdots$$

The meniscus solution suggests $H = \frac{1}{\sqrt{2}} \epsilon^2 \bar{z}^2$. Therefore, set $\underbrace{H = \epsilon^2 \bar{H}} + \epsilon^4 \hat{H} + \cdots$

$$\bar{H}(\bar{z}) = H/\epsilon^2 \;,\; \bar{z} = \frac{1}{\epsilon}(z - z_0) \;,\; \frac{\partial}{\partial z} = \frac{\partial}{\partial \bar{z}} \cdot \frac{\partial \bar{z}}{\partial z} = \frac{\partial}{\partial \bar{z}} \frac{1}{\epsilon}$$

The steady state equation becomes

$$\frac{\epsilon^2}{\epsilon} \bar{H}_{\bar{z}} + \frac{1}{\epsilon}\left[\frac{1}{3Ca} \epsilon^6 \bar{H}^3 \left(\frac{\epsilon^2}{\epsilon^3} \bar{H}_{\bar{z}\bar{z}\bar{z}} - 1 \right) \right]_{\bar{z}} = 0$$

$$\rightarrow \bar{H}_{\bar{z}} + \left[\frac{\epsilon^3}{3Ca} \bar{H}^3 \left(\bar{H}_{\bar{z}\bar{z}\bar{z}} - \epsilon \right) \right]_{\bar{z}} = 0$$

For $Ca \ll 1$, surface tension is important in the dynamics. So take $\epsilon = Ca^{1/3} \ll 1$ and have

$$\bar{H}_{\bar{z}} + \left[\frac{1}{3} \bar{H}^3 \left(\bar{H}_{\bar{z}\bar{z}\bar{z}} - \epsilon \right) \right]_{\bar{z}} = 0$$

demonstrating that gravity is not important in the thin film regoin at leading order.

At leading order $O(\epsilon^0)$, we have

$$\bar{H}_{\bar{z}} + \frac{1}{3}\left(\bar{H}^3 \bar{H}_{\bar{z}\bar{z}\bar{z}} \right)_{\bar{z}} = 0 \,, \qquad \circledast$$

subject to

$$\bar{H} \to \bar{H}_0 = H_0 \epsilon^{-2} = \frac{h_0}{l_c} Ca^{-2/3} \sim O(1) \quad \text{as } \bar{z} \to \infty$$

$$\bar{H} \sim \frac{1}{\sqrt{2}} \bar{z}^2 \quad \text{as } \bar{z} \to -\infty$$

where $\bar{H}_0$ is to be determined. Integrating $\circledast$ once gives

$$\boxed{\bar{H} + \frac{1}{3}\bar{H}^3 \bar{H}_{\bar{z}\bar{z}\bar{z}} = \bar{H}_0}$$

We know $\bar{H}_0$ is not arbitrary. Instead, there is a specific choice of $\bar{H}_0$ so that the behavior is matched as $\bar{z} \to -\infty$. What is it?

- Examination of the behavior as $\bar{z} \to \infty$

Linearization: $\bar{H} = \bar{H}_0 + f$ with $|f| \ll \bar{H}_0$

$$\to f + \frac{1}{3}\bar{H}_0^3 f_{\bar{z}\bar{z}\bar{z}} = 0$$

Seek solutions in the form $f = e^{\lambda \bar{z}}$

$$\underbrace{1}_{-e^{i(\pi+2n\pi)},\ n=0,1,2\cdots} + \frac{1}{3}\bar{H}_0^3 \lambda^3 = 0 \quad , \quad \lambda = \frac{3^{1/3}}{\bar{H}_0} e^{i\pi/3},\ -\frac{3^{1/3}}{\bar{H}_0},\ \frac{3^{1/3}}{\bar{H}_0} e^{-i\pi/3}$$

$$\to f = A\exp\left[-3^{1/3}\bar{z}/\bar{H}_0\right] + \underbrace{B\exp\left[3^{1/3}e^{i\pi/3}\bar{z}/\bar{H}_0\right] + C\exp\left[3^{1/3}e^{-i\pi/3}\bar{z}/\bar{H}_0\right]}_{\text{Real part} > 0 \text{ so } B=C=0 \text{ to satisfy conditions as } \bar{z}\to\infty}$$

- Examination of the behavior as $\bar{z} \to -\infty$

Linearization: $\bar{H} = \frac{1}{\sqrt{2}}\bar{z}^2 + f$ with $|f| \ll \frac{1}{\sqrt{2}}\bar{z}^2$

$$\frac{1}{\sqrt{2}}\bar{z}^2 + \underbrace{\cancel{f}}_{\ll \bar{z}^2} + \frac{1}{6\sqrt{2}}\bar{z}^6 f_{\bar{z}\bar{z}\bar{z}} = \bar{H}_0 \to f_{\bar{z}\bar{z}\bar{z}} = \frac{6\sqrt{2}\bar{H}_0}{\bar{z}^6} - \frac{6}{\bar{z}^4} \sim -\frac{6}{\bar{z}^4}$$

Solution takes $f \sim \underbrace{\cancel{a}}_{0 \text{ since } f \ll \bar{z}^2}\bar{z}^2 + \underbrace{b\bar{z} + c}_{\text{Arbitrary}} + \frac{1}{\bar{z}}$

• The system is translation-invariant.

Fix the origin removes a degree of freedom. This is equivalent to fixing the coefficient A. Let $A = \bar{H}_0$ - we are particularly interested in the behavior at $\pm\infty$.

Now rescale the ode by $\bar{H} = \bar{H}_0 g$, $\bar{Z} = \bar{H}_0 \xi$ and seek solution to

$$g + \frac{1}{3} g^3 g_{zzz} = 1$$

$$g \sim 1 + e^{-3^{1/3}\xi} \quad \text{as } \xi \to +\infty$$

$$g \sim \frac{\bar{H}_0}{\sqrt{2}} \xi^2 \quad \text{as } \xi \to -\infty$$

Note that as $g \to \infty$ ($\xi \to -\infty$), g_{zzz} has to go to 0, i.e. $g \propto \xi^2$. Numericall shooting from infinity we find

$$g \sim 0.67 \xi^2 \quad \text{as } \xi \to -\infty$$

Thus. $\bar{H}_0 = 0.67 \times \sqrt{2} = 0.948$, i.e.,

$$h_0 = 0.948\, l_c\, Ca^{2/3} = 0.948 \left(\frac{\gamma}{\rho g}\right)^{1/2} \left(\frac{\mu U}{\gamma}\right)^{2/3} = 0.948 \frac{\mu^{2/3} U^{2/3}}{\gamma^{1/6} \rho^{1/2} g^{1/2}}$$

• Silicone oil: $\rho g \sim 8000\ N/m^3$, $\gamma = 20\ mJ/m^2$, $\mu = 10^{-2}\ Pa\cdot s$, $U = 1\ mm/s$

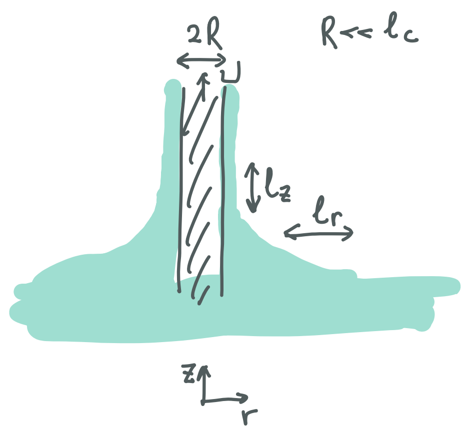
$l_c \sim 1.6mm$, $Ca \sim 10^{-4}$, $h_0 \sim 10\ \mu m$

- Jump out of pool: $\rho g = 9800\ N/m^3$, $\gamma = 72\ mJ/m^2$, $\mu = 10^{-3}\ Pa\cdot s$, $U = 1\ m/s$

$l_c \sim 2.7mm$, $Ca \sim 10^{-2}$, $h_0 \sim 0.15\ mm$

Other examples

① Withdrawing a fiber from a bath.



$R \ll l_c$

Shear stress "gradient" $\sim \mu \frac{U}{h_0^2}$

Capillary pressure "gradient" $\sim \gamma \frac{h_0}{l^3}$

Curvature of static meniscus $\sim \frac{1}{R} + \frac{1}{l_c} \sim \frac{1}{R}$

Matching κ $\Rightarrow$ $\frac{h_0}{l^2} \sim \frac{1}{R} \rightarrow l \sim (h_0 R)^{1/2}$

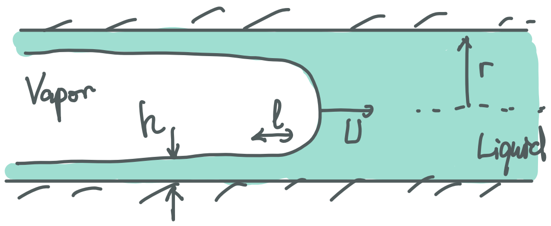
$$\mu \frac{U}{h_0^2} \sim \gamma \frac{h_0}{l^3} \Rightarrow h_0 \sim R \left(\frac{\mu U}{\gamma}\right)^{2/3}$$

$$\boxed{\Rightarrow h_0 = \begin{cases} 0.95\, l_c\, Ca^{2/3}, & \text{for plates} \\ 1.34\, R\, Ca^{2/3}, & \text{for fibers, where } R \ll l_c \end{cases}}$$

Landau-Levich.

② Displacement of an interface in a tube

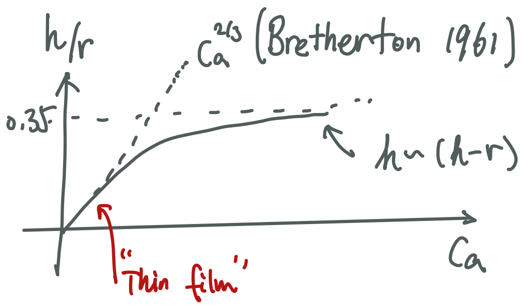
Air evacuating a water-filled pipette or pumping oil out of rock with water



$h \ll r \ll l_c$

- $\nabla p \sim \gamma \times \frac{1}{r} \times \frac{1}{l} = \frac{\gamma}{rl}$
- $\frac{1}{r} + \frac{h}{l^2} \sim \frac{2}{r} \rightarrow l \sim (hr)^{1/2}$
- $\mu \frac{\partial^2 u}{\partial r^2} \sim \frac{\mu U}{h_0^2}$

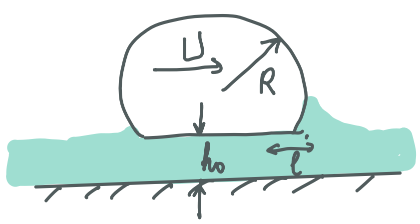
$$\Rightarrow h_0 \sim r\, Ca^{2/3}$$



$h \sim (h-r)\, Ca^{2/3}$ for $h \sim r$ (Taylor 1961)

"Thick film"

③ Drop moving a liquid-lubricated surfaces



In Daniel et al. Nat. Phys. (2017), it is found

$$h_0 \sim R\, Ca^{2/3}$$

The force needed is calculated by assuming dissipation mostly occuring at the rim of length l.

$$F \sim 2\pi R l \times \tau_s$$

$$\tau_s \sim \mu \frac{U}{h_0}$$

$$\rightarrow F \sim \frac{2\pi \mu U R l}{h_0} = \frac{2\pi \mu U R \cdot R Ca^{1/3}}{R\, Ca^{2/3}} \quad \leftarrow l \sim \sqrt{R h_0} \sim R Ca^{1/3}$$

$$= 2\pi \gamma R\, Ca^{2/3}$$