Fluid dynamics

• Thin film dewetting ($Re \ll 1$)

In previous lectures, we have discussed the shape of a sessile drop.

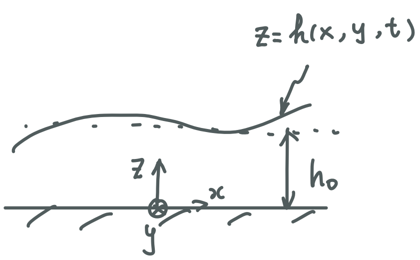
Now the questions arise for a thin film dewetting into drops: When / How / How fast?

- When does it occur?
- What is the characteristic size of the dewetting regions?
- How fast does it happen?

To answer these questions, first need a dynamics equation for h.

Kinematics

The surface satisfies $F(x,y,t) = z - h(x,y,t) = 0$



$\frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \nabla$ material derivative

$$\frac{dF}{dt} = \underbrace{\frac{dz}{dt}\Big|_s}_{w_s} \overbrace{- \frac{\partial h}{\partial x} \underbrace{\frac{dx}{dt}\Big|_s}_{u_s} - \frac{\partial h}{\partial y} \underbrace{\frac{dy}{dt}\Big|_s}_{v_s} - \frac{\partial h}{\partial t}} = 0$$

$$\boxed{w_s = \frac{\partial h}{\partial t} + u_s \frac{\partial h}{\partial x} + v_s \frac{\partial h}{\partial y}}$$ Kinematic boundary condition.

Conservation of mass

Incompressibility : $\nabla \cdot \underset{\sim}{u} = 0$, $\quad \underset{\sim}{u} = u\,\underset{\sim}{e}_x + v\,\underset{\sim}{e}_y + w\,\underset{\sim}{e}_z$

$$\int_0^{h(x,y,t)} \nabla \cdot \underset{\sim}{u}\, dz = \int_0^h \frac{\partial u}{\partial x} dz + \int_0^h \frac{\partial v}{\partial y} dz + \underbrace{\int_0^h \frac{\partial w}{\partial z} dz}_{= w_s - \cancel{w(0)}^{\,0}}$$

Recall Leibniz' therom

$$\frac{\partial}{\partial x}\int_0^h u\, dz = \int_0^h \frac{\partial u}{\partial x} dz + u \frac{\partial h}{\partial x}\bigg|_{z=h} = \int_0^h \frac{\partial u}{\partial x} dz + u_s \frac{\partial h}{\partial x}$$

Similarly, we have

$$\frac{\partial}{\partial y}\int_0^h v\, dz = \int_0^h \frac{\partial v}{\partial y} dz + v_s \frac{\partial h}{\partial y}$$

Combining these leads to

$$\frac{\partial}{\partial x}\int_0^h u\, dz - u_s \frac{\partial h}{\partial x} + \frac{\partial}{\partial y}\int_0^h v\, dz - v_s \frac{\partial h}{\partial y} + w_s = 0$$

Use the kinematic boundary condition to show

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h u\, dz + \frac{\partial}{\partial y}\int_0^h v\, dz = 0$$

Or

$$\boxed{\frac{\partial h}{\partial t} + \nabla \cdot \int_0^h \underset{\sim}{u}\, dz = 0}$$ Continuity condition.

flux Q

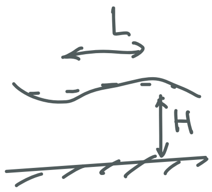
Conservation of momentum

Navier-stokes equation for an incompressible Newtonian viscous fluid.

$$\underbrace{\rho\frac{d\underline{u}}{dt} = \rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right)}_{\text{Inertia}} = \overbrace{-\nabla p + \underbrace{\mu\nabla^2\underline{u}}_{\text{Diffusion}}}^{\text{Divergence of stress}} + \rho\underline{g}$$

gravitational acceleration

Before proceeding, let us disscuss the scaling due to the thin geometry.

-  $\Rightarrow \frac{\partial}{\partial x} \sim \frac{1}{L} \ll \frac{\partial}{\partial z} \sim \frac{1}{H} \rightarrow \epsilon = \frac{H}{L}$ (small parameter)

- $\nabla\cdot\underline{u} = 0 \rightarrow u \sim v \sim \frac{W}{\epsilon}$

- $\rho(\underline{u}\cdot\nabla\underline{u}) \sim \rho u^2/L$, $\mu\nabla^2\underline{u} \sim \mu u/H^2$. Inertia may be neglected as long as $Re^r = \frac{\rho u L}{\mu} \times \frac{H^2}{L^2} = Re\cdot\epsilon^2 \ll 1$ (Reduced Reynolds number). Inertia may be neglected even though Re is not that small.

We then consider the stokes equations

$$\mu\left(\underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}_{\epsilon^2[u]/H^2} + \underbrace{\frac{\partial^2 u}{\partial z^2}}_{[u]/H^2}\right) - \frac{\partial p}{\partial x} = 0$$

$$\mu\left(\underbrace{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}}_{\epsilon^2[u]/H^2} + \underbrace{\frac{\partial^2 v}{\partial z^2}}_{[u]/H^2}\right) - \frac{\partial p}{\partial y} = 0$$

$$\mu\left(\underbrace{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}}_{\epsilon^3[u]/H^2} + \underbrace{\frac{\partial^2 w}{\partial z^2}}_{\epsilon[u]/H^2}\right) - \underbrace{\frac{\partial p}{\partial z}}_{\frac{\partial p}{\partial x}\frac{1}{\epsilon} \sim [u]/H^2\cdot\frac{1}{\epsilon}} + \rho g = 0$$

We focus on long-wavelength limit $\epsilon = \frac{H}{L} \ll 1$. The flow is mostly unidirectional (a.k.a. lubrication approximation)

$$\begin{cases} \mu \frac{\partial^2 u}{\partial z^2} - \frac{\partial p}{\partial x} = 0 \\ \mu \frac{\partial^2 v}{\partial z^2} - \frac{\partial p}{\partial y} = 0 \\ \frac{\partial p}{\partial z} = \rho g \rightarrow p = \rho g (z - z_0) + f(x,y) \end{cases} \Rightarrow \mu \frac{\partial \underset{\sim}{u}}{\partial z^2} - \nabla_s p = 0, \quad \underset{\sim}{u} = u \underset{\sim}{e}_x + v \underset{\sim}{e}_y, \quad \nabla_s = \frac{\partial}{\partial x}\underset{\sim}{e}_x + \frac{\partial}{\partial y}\underset{\sim}{e}_y$$

The pressure is approximately hydrostatic; The flow is quasi-parallel to the boundary

$$\underset{\sim}{u}(z) = \frac{1}{2\mu} \nabla_s p \, z^2 + \underset{\sim}{c}_1 z + \underset{\sim}{c}_2$$

Boundary conditions.

① $\underset{\sim}{u}(0) = 0$ (No-slip) $\rightarrow \underset{\sim}{c}_2 = 0$

② $\frac{\partial \underset{\sim}{u}}{\partial z}\Big|_h = 0$ (No-shear) $\rightarrow \underset{\sim}{u}(z) = \frac{1}{2\mu} \nabla p \, (z^2 - 2zh)$

Flux across the thickness $\underset{\sim}{Q} = \int_0^h \underset{\sim}{u}(z)\, dz = -\frac{h^3}{3\mu} \nabla p$

Conservation of mass $\frac{\partial h}{\partial t} + \nabla \cdot \underset{\sim}{Q} = 0$ gives

$$\boxed{\frac{\partial h}{\partial t} = \frac{1}{3\mu} \nabla \cdot \left(h^3 \nabla p\right)}$$ Reynold's equation.

Aside: What if ϵ is not that small?

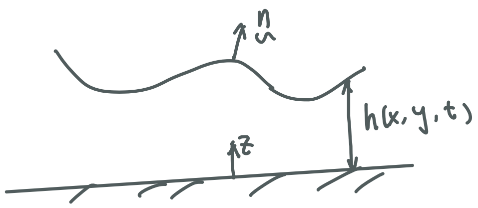
$$\underset{\sim}{u}(\underset{\sim}{x}, \epsilon) = \underset{\sim}{u}_0(\underset{\sim}{x}) + \epsilon^2 \underset{\sim}{u}_2(\underset{\sim}{x}) + \epsilon^4 \underset{\sim}{u}_4(\underset{\sim}{x}) + O(\epsilon^6)$$

$$w(\underset{\sim}{x}, \epsilon) = w(\underset{\sim}{x}) + \epsilon^2 w_2(\underset{\sim}{x}) + \epsilon^4 w_4(\underset{\sim}{x}) + O(\epsilon^6)$$

$$p(\underset{\sim}{x}, \epsilon) = p_0(\underset{\sim}{x}) + \epsilon^2 p_2(\underset{\sim}{x}) + \epsilon^4 p_4(\underset{\sim}{x}) + O(\epsilon^6)$$

Our focus has been on the leading order result $O(\epsilon^0)$

Typical source of pressure gradients



$$p = \gamma \nabla \cdot \underset{\sim}{n} + \frac{A}{6\pi h^3} + \rho g (h - z_0) + f(h) \ \cdots$$

electric

reference.

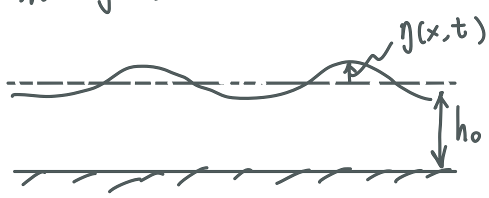
In particular, $\nabla \cdot \underset{\sim}{n} = \frac{\nabla F}{|\nabla F|} = \frac{-(h_{xx} + h_{yy})}{(1 + h_x^2 + h_y^2)^{3/2}} \simeq -\nabla^2 h$ (Lubrication approximation)

Constants

$$\boxed{\nabla p = -\gamma \nabla(\nabla^2 h) - \frac{A}{2\pi h^4} \nabla h + \rho g \nabla h}$$

1D thin film dewetting (Neglecting gravity)

$h(x,y,t) = h(x,t)$



$$\frac{\partial h}{\partial t} = \frac{1}{3\mu} \frac{\partial}{\partial x} \left[h^3 \left(-\gamma \frac{\partial^3 h}{\partial x^3} - \frac{A}{2\pi h^4} \frac{\partial h}{\partial x} \right) \right]$$

$$= \frac{1}{3\mu} \frac{\partial}{\partial x} \left(-\gamma h^3 \frac{\partial^3 h}{\partial x^3} - \frac{A}{2\pi h} \frac{\partial h}{\partial x} \right)$$

In statics, $\frac{\partial h}{\partial t} = 0$. In dynamics $\frac{\partial h}{\partial t} \neq 0$, whether a perturbation grows or decay?

(<u>linear</u> stability analysis)

Consider $h = h_0 + \eta(x,t)$, $\frac{|\eta|}{h_0} \ll 1$

- $\frac{\partial h}{\partial t} = \frac{\partial \eta}{\partial t}$

- $h^3 = (h_0 + \eta)^3 = h_0^3\left(1 + \frac{3\eta}{h_0} + \cdots\right)$

- $h^3 \frac{\partial^2 h}{\partial x^2} = h_0^3 \frac{\partial^2 \eta}{\partial x^2} + \cancelto{0}{3\frac{\eta}{h_0} h_0^3 \frac{\partial^2 \eta}{\partial x^2}}$

- $\frac{1}{h}\frac{\partial h}{\partial x} = \frac{1}{h_0}\frac{\partial \eta}{\partial x} - \cancelto{0}{\frac{\eta}{h_0}\frac{1}{h_0}\frac{\partial \eta}{\partial x}}$

We obtain linearized form of govering equation

$$\frac{\partial \eta}{\partial t} = -\frac{\gamma h_0^3}{3\mu}\frac{\partial^4 \eta}{\partial x^4} - \frac{A}{6\pi\mu h_0}\frac{\partial^2 \eta}{\partial x^2}$$

Since this linearized equation has coefficients INdependent of x and t, seek

separable solution of form

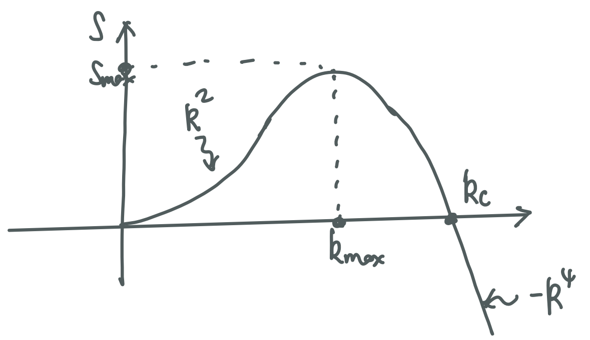
$$\eta = \eta_0 e^{st} e^{ikx}$$

wavenumber

so that the amplitude decays if $Re(s) < 0$ and grows if $Re(s) > 0$.

The characteristic equation is

$$s = -\frac{\gamma h_0^3}{3\mu} k^4 + \frac{A}{6\pi\mu h_0} k^2 \quad \text{(Always real number)}$$



- $S > 0$ is possible if $A > 0$
- Surface tension stablizes the systems ($s<0$)
- There is a fastest growing mode ($A>0$)

☆ $s<0$ for $k>k_c$

$$k_c = \left(\frac{A}{2\pi\gamma}\right)^{1/2}\frac{1}{h_0^2}$$ ← when to occur? Depends on h_0 and system size L.

☆ $$\left.\frac{ds}{dk}\right|_{k_{max}} = -\frac{4}{3}\frac{\gamma h_0^3}{\mu}k_{max}^3 + \frac{A}{3\pi\mu h_0}k_{max} = 0$$

$$k_{max} = \left(\frac{A}{4\pi\gamma}\right)^{1/2}\frac{1}{h_0^2} = \frac{1}{\sqrt{2}}k_c$$

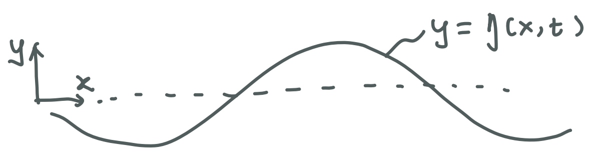
$$s_{max} = \frac{A^2}{48\pi^2\gamma\mu h_0^5}$$

$A = 10^{-20}\,J$, $\gamma = 20\ mJ/m^2$, $h_0 = 100\ nm$ → $\lambda_{max} = \frac{2\pi}{k} \simeq 31.5\ \mu m$

$1000\ nm$ → $31.5\ mm$

$10\ \mu m$ → $3\,m \gg l_c$

Capillary and gravity wave ($Re \gg 1$)



How y behaves?

$F(x,y,t) = y - \eta(x,t)$

Now consider a 2D inviscid flow with infinite depth

Kinematics

$$\frac{dF}{dt} = \underset{\uparrow\, v_s}{\frac{dy}{dt}} - \frac{\partial\eta}{\partial x}\underset{\uparrow\, u_s}{\frac{dx}{dt}} - \frac{\partial\eta}{\partial t} = 0$$ (on the surface)

→ $v_s = \frac{\partial\eta}{\partial t} + u_s\frac{\partial\eta}{\partial x}$ or $v_\perp = \frac{\partial\eta}{\partial t} + \underset{\sim}{u}_{\parallel}\nabla\eta$

Continuity

$$\nabla \cdot \underset{\sim}{u} = 0 \rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Momentum

$$\frac{d\underset{\sim}{u}}{dt} = \frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u}$$

True pressure

$$\rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u}\right) = -\nabla p_T + \rho \underbrace{\underset{\sim}{g}}_{-\underset{\sim}{e}_y} + \cancel{\mu \nabla^2 \underset{\sim}{u}} \rightarrow 0$$

(Unsteady inviscid flows) Euler equation. $\mu \to 0$

Let $p = p_T + \rho g y$ and obtain

$$\rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u}\right) = -\nabla p$$ dynamic pressure

Linearization

Consider small amplitude so that η and $\underset{\sim}{u}$ ($u \sim \frac{\partial \eta}{\partial t}$) are small, so is $\underset{\sim}{u} \cdot \nabla \underset{\sim}{u}$

We then have linearized governing equations:

$$\nabla \cdot \underset{\sim}{u} = 0$$

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} = -\nabla p$$

$$\nabla \cdot \left(\rho \frac{\partial \underset{\sim}{u}}{\partial t}\right) = \rho \frac{\partial \nabla \cdot u}{\partial t} = 0 = -\nabla^2 p$$

$$\rightarrow \nabla^2 p = 0$$

Assume $\eta(x,t) = \hat{\eta} \exp[i k(x - \overset{\text{wave speed}}{c} t)]$

$$= \hat{\eta} \exp[i(\underset{\text{wavenumber}}{k} x - \underset{\text{frequency}}{\omega} t)]$$

$k = \frac{2\pi}{\lambda}$ $\quad c = \frac{\omega}{k}$

It's like taking $s = -i\omega t$ - Never decay!! It's fine to be wrong, then we will end up with a complex ω

Then natural to assume

$$P(x,y,t) = \underset{\uparrow}{\hat{p}(y)} \exp[i(kx-\omega t)]$$

The simplest guess – base pressure difference is more interesting in y direction than x direction!

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \underbrace{\left(-k^2 \hat{p} + \frac{d^2\hat{p}}{dy^2}\right)}_{=0} \exp[i(kx-\omega t)] = 0$$

The solution is $\hat{p} = \underset{\swarrow 0}{\cancel{Ae^{-ky}}} + Be^{ky}$

(Infinity $\to 0$ since $\hat{p} = \hat{p}_T + \rho g y$ already considered gravity)

$$\to \ p = Be^{ky} e^{i(kx-\omega t)} + \text{Any constants (eg. } P_{atm})$$

Now back to the momentum equation (dynamic boundary condition)

$$\rho \frac{\partial u}{\partial t} = -\nabla P \to \rho \frac{\partial v}{\partial t} = -Bke^{ky} e^{i(kx-\omega t)}$$

$$\boxed{v = \frac{-iBk}{\rho\omega} e^{ky} e^{i(kx-\omega t)}}$$

Kinematic B.C. $\to$ $v_s = \frac{\partial \eta}{\partial t} + O(\eta^2) = -i\omega\hat{\eta} e^{i(kx-\omega t)}$ (On the surface)

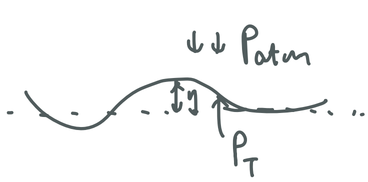
We then have

$$\omega\hat{\eta} = \frac{Bk}{\rho\omega} e^{ky}\Big|_{y=\eta} = \frac{Bk}{\rho\omega}\left(1 + \cancel{k\eta} + \cdots\right)$$

$\cancel{k\eta} \to 0$ Neglect this to be consistent.

$$\Rightarrow \boxed{B = \frac{\rho\omega^2\hat{\eta}}{k}}$$ This is not done since $\hat{\eta}$ remains unknow.

We haven't used pressure jump condition on the surface (stress b.c.)



$$P_T = P_{atm} + \gamma \underbrace{\nabla \cdot \underset{\frown}{n}}_{\leftarrow \simeq -\frac{\partial^2\eta}{\partial x^2}} = P(y=\eta) - \rho g \eta$$

$$\rightarrow \gamma \hat{\eta} k^2 e^{i(kx-\omega t)} = \beta e^{\cancel{k\eta}} e^{i(kx-\omega t)} - \rho g \hat{\eta} e^{i(kx-\omega t)}$$

$$\rightarrow \cancel{\gamma \hat{\eta}} k^2 = \frac{\rho \omega^2 \cancel{\hat{\eta}}}{k} - \rho g \cancel{\hat{\eta}}$$ ($\hat{\eta}$ does not matter – linearization is successfull)

We finally have

$$\omega = \left(\frac{\gamma k^3}{\rho} + g k \right)^{1/2}$$

← Dispersion relation 色散关系

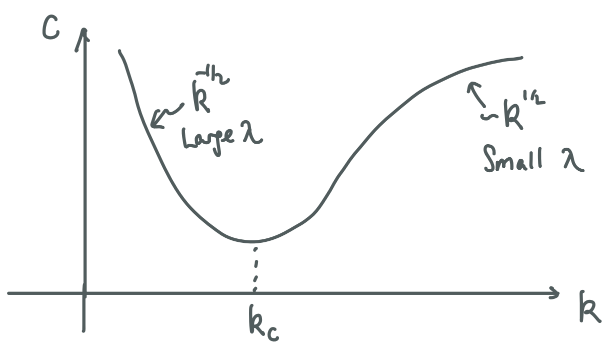
A wave is called dispersive since its different Fourier components disparate / disperse.

and

$$c = \left(\frac{\gamma k}{\rho} + \frac{g}{k} \right)^{1/2}$$

← In a dispersive system, the energy propogates at the group speed $c_g = \frac{d\omega}{dk}$ instead of phase speed $c = \frac{\omega}{k}$

Observations



① Relative importance of γ and g is described by Bond number

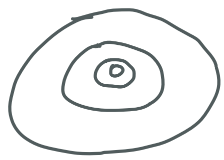
$$Bo = \frac{\rho g}{k^2 \gamma} = \frac{\rho g \lambda^2}{4\pi^2 \gamma}.$$

For air–water, $Bo \sim 1$ for $\lambda = 2\pi l_c \sim 1.7$ cm

② $Bo >> 1$ as $\lambda >> 2\pi l_c$: Surface tension effect negligible ⇒ Gravity wave

$$c \simeq (g/k)^{1/2} = (g\lambda/2\pi)^{1/2}$$

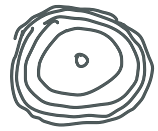
Longer waves travel faster



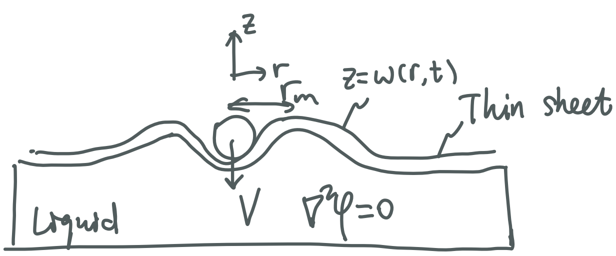
③ $Bo \ll 1$ as $\lambda \ll 2\pi l_c$: Influence of g negligible $\Rightarrow$ Capillary wave

$$c \simeq (\gamma k/\rho)^{1/2} = (2\pi\gamma/\rho\lambda)^{1/2}$$

Short waves travel fastest !



· Elasto-capillary waves



Inviscid, incompressible, irrotational.

Now take $\underset{\sim}{u} = \nabla\varphi \leftarrow$ velocity potential

- Incompressibility $\nabla\cdot\underset{\sim}{u}=0$, requiring

$$\nabla^2\varphi = 0$$

- Kinematic condition

$$\underbrace{\frac{\partial\varphi}{\partial z}}_{u_z} = \frac{\partial w}{\partial t} + \underbrace{\frac{\partial\varphi}{\partial r}}_{u_r}\frac{\partial w}{\partial r}$$

- Dynamic boundary condition

$$\rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u}\right) = -\nabla p + \rho\underset{\sim}{g} \longrightarrow \nabla\left[\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho|\nabla\varphi|^2 + p - \rho g z\right] = 0$$

$$\underset{\sim}{u}\cdot\nabla\underset{\sim}{u} = \frac{1}{2}\nabla\left[u^2 - \underset{\sim}{u}\times\cancel{(\nabla\times\underset{\sim}{u})}\right], \quad \nabla\times\underset{\sim}{u} = 0$$

Irrotational flow

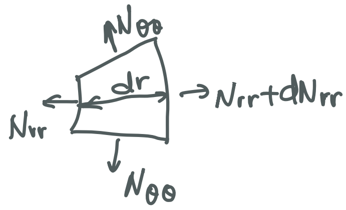
$$\rightarrow \rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho|\nabla\varphi|^2 + p - \rho g = f(t)$$

(Bernoulli's equation)

- Stress boundary condition

① Capillary force only $p \simeq -\gamma \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right)$

② Membrane force only $p = - N_{rr} \frac{\partial^2 w}{\partial r^2} - N_{\theta\theta} \frac{1}{r} \frac{\partial w}{\partial r}$



$$\frac{\partial N_{rr}}{\partial r} + \frac{N_{rr} - N_{\theta\theta}}{r} = 0$$

Relaxed membrane $N_{\theta\theta} \ll N_{rr} \rightarrow N_{rr} = \frac{c}{r} = \frac{\gamma R_f}{r}$ $\left(\epsilon_{rr} \sim \frac{N_{rr}}{Y} = \frac{\gamma R_f}{Y r} \sim 10^{-4} \right)$

Tensioned membrane $N_{rr} \sim Y \epsilon_{rr} \sim Y \cdot \left(\underbrace{\frac{\rho R_s V^2}{Y}}_{We} \right)^{1/3}$ $\left(\epsilon_{rr} \sim We^{1/3} \sim 1 \right)$

- Scaling point of view

Mass: $\nabla^2 \varphi = 0 \rightarrow z_* \sim r_m$ (The base feels the r_m in the way it feels z_*)

Momentum: $\varphi_* \sim V \times z_* \sim V \times r_m$

Stress: $\rho \frac{\varphi_*}{t} \sim p \sim \begin{cases} \gamma \frac{Vt}{r_m^2} \text{ (capillary force)} \rightarrow \varphi_* \sim \frac{\gamma V t^2}{\rho r_m^2} \\ \frac{\gamma R_f}{r_m} \frac{Vt}{r_m^2} \text{ (Relaxed membrane)} \rightarrow \varphi_* \sim \frac{\gamma R_f V t^2}{\rho r_m^3} \\ Y We^{1/3} \frac{Vt}{r_m^2} \text{ (Tensioned membrane)} \rightarrow \varphi_* \sim \frac{Y We^{1/3} V t^2}{\rho r_m^2} \end{cases}$

$$\Rightarrow \begin{cases} \text{Inertia-Capillary}: r_m \sim \left(\frac{\gamma}{\rho} \right)^{1/3} t^{2/3} \\ \text{Inertia-elasto-capillary (Relaxed)}: r_m \sim \left(\frac{\gamma R_f}{\rho} \right)^{1/4} t^{1/2} \\ \text{Inertia-elasto-capillary (Tensioned)}: r_m \sim \left(\frac{Y}{\rho} \right)^{1/3} \left(\frac{\rho R_s V^2}{Y} \right)^{1/9} t^{2/3} \end{cases}$$