Lateral forces

Today we will end the discussion of Fluid Statics by discussing Cheerios effect and it's different version in which lateral forces are important.

We have discussed Buoyancy, Surface tension, and vdW forces.

① Buoyance + Surface tension

$\sim \rho g h$ $\quad\quad \sim \gamma \frac{h}{\ell^2}$ $\quad\quad$ Define $\ell_c = \left(\frac{\gamma}{\rho g}\right)^{1/2} \rightarrow B_o = \left(\frac{L}{\ell_c}\right)^2 = \begin{cases} \gg 1, & \text{G-dom.} \\ \sim O(1), & ☆ \text{ (Today)} \\ \ll 1, & \text{C-dom} \end{cases}$

size of the system (L)

② VdW forces + Surface tenion

$\sim \frac{A}{h^3}$ $\quad\quad \sim \gamma \frac{h}{\ell^2}$ $\quad\quad \ell_{vdw} \sim \left(\frac{\gamma h_c^4}{A}\right)^{1/2} \rightarrow V_\gamma = \left(\frac{L}{L_{vdw}}\right)^2 \sim O(1)$ ☆ (Today)

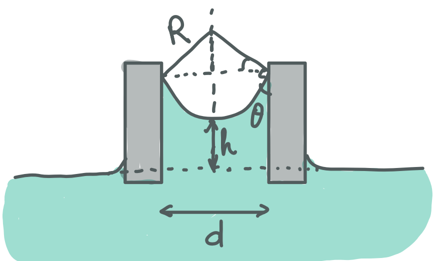
Size of the system (L)

[Note we have neglected gravity, requiring $h \ll \left(\frac{A}{\rho g}\right)^{1/4}$ or $\ell_{vdw} \ll \ell_c$]

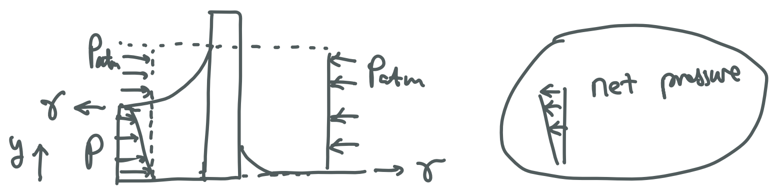
• Lateral capillary forces

• First, consider a simple model where d is small (what is meant by "small"?)

Cut $\quad R = \frac{d}{2\cos\theta}$



→ Positive interaction



$P < P_{atm}$ → Attraction between two plates

Determine h: $\quad P_{atm} - \frac{\gamma}{R} + \rho g h = P_{atm} \rightarrow h = \frac{\gamma}{\rho g R} = \frac{2\gamma\cos\theta}{\rho g d}$

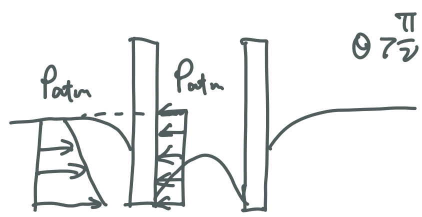
Compute plate-plate interaction:

$$\frac{F}{W} = -\int_0^h \rho g y\, dy = -\frac{1}{2}\rho g h^2 = -\frac{2\gamma^2\cos^2\theta}{\rho g d^2} = -2\gamma\left(\frac{l_c}{d}\right)^2\cos^2\theta$$

width of plate

Note that ① this is only true for $d \ll l_c$!

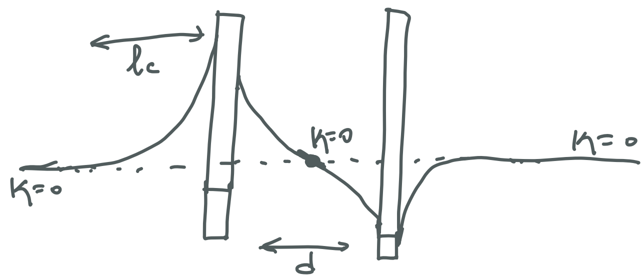
② The θ term is $\cos^2\theta \geq 0$ (Always attractive interaction for $\theta \neq \frac{\pi}{2}$)



③ May guess, for plates with different surface energies

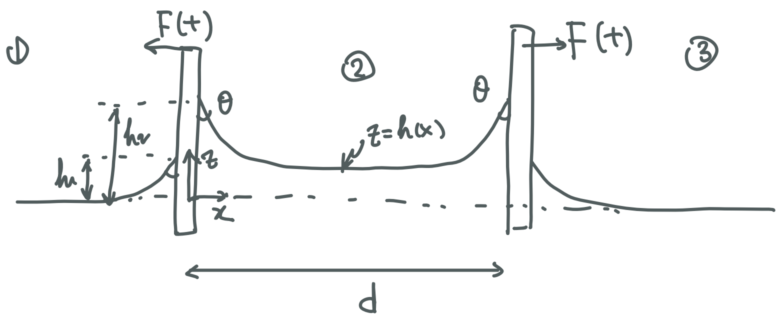
$$\frac{F}{W} = -2\left(\frac{l_c}{d}\right)^2 \cos\theta_1 \times \cos\theta_2\ , \ ?$$

④ Then the interaction could be repulsive, say $\theta_1 = 0$, $\theta_2 = \pi$



what if $d \ll l_c$?

- We then consider a slightly formal analysis for far-field interactions.



$$\frac{F}{W} = -\frac{1}{2}\rho g\left(h_2^2 - h_1^2\right)$$

Line forces are cancelled!

For both ① and ②, we have $\gamma \nabla\cdot\hat{n} + \rho g h = 0$, where $\nabla\cdot\hat{n} = \dfrac{-h_{xx}}{(1+h_x^2)^{3/2}}$

Consider $|h_x^2| \ll 1$, we have

$$\gamma h_{xx} - \rho g h = 0$$

or

$$h_{xx} - \frac{1}{l_c^2} h = 0 \,,$$

to which the solution is $h_i(x) = A_i e^{+x/l_c} + B_i e^{-x/l_c}$.

- For ①, $h(-\infty) = 0 \to B_1 = 0$, $h_x(0) = \cot\theta \to A_1/l_c = \cot\theta$ $\Big\} \to h_1(x) = l_c \cot\theta \, e^{x/l_c}$

- For ② $h_x(0) = -\cot\theta$, $h_x(d) = \cot\theta$ $\Big\} \to h_2(x) = l_c \cot\theta \dfrac{\cosh\left(\frac{d-x}{l_c}\right) + \cosh\left(\frac{x}{l_c}\right)}{\sinh(d/l_c)}$

Therefore,

$$\frac{F}{W} = -\frac{1}{2}\rho g \left[h_2^2(0) - h_1^2(0) \right] = -\frac{\gamma}{2}\cot^2\theta \left[\frac{\left(\cosh(d/l_c)+1\right)^2}{\sinh^2(d/l_c)} - 1 \right] = \underbrace{-\frac{\gamma}{2}\cot^2\theta / \sinh^2(d/2l_c)}_{= -\frac{1}{2}\rho g h^2\left(\frac{d}{2}\right)}$$

- Consider $d \ll l_c$ $\left(\cosh x \to 1 + \frac{1}{2}x^2 , \ \sinh x \to x\right)$

$$\frac{F}{W} = -\frac{\gamma}{2}\cot^2\theta \, \frac{4}{d^2/l_c^2} = -\frac{2\gamma^2 \cos^2\theta}{\rho g d^2 \sin^2\theta}$$

d^{-2} decay (Same to vdW ?)

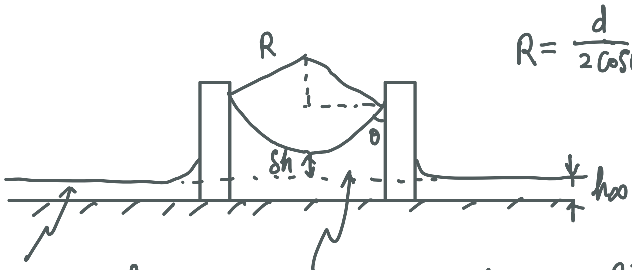
- Conside $d \gg l_c$

$$\frac{F}{W} = -\frac{\gamma}{2}\frac{\cot^2\theta}{\sinh^2(d/2l_c)} \simeq -\frac{2\gamma\cot^2\theta}{e^{d/l_c}}$$

Exponential decay.

• Lateral immersion forces

Now neglect buoyancy / gravitational forces and consider a limiting case "small d"



$$R = \frac{d}{2\cos\theta}$$

$$P = P_{atm} + \frac{A_{SLV}}{6\pi h_\infty^3} \qquad P = P_{atm} + \frac{A_{SLV}}{6\pi h^3} - \frac{2\gamma\cos\theta}{d}$$

$$\rightarrow \frac{A_{SLV}}{6\pi}\left(\frac{1}{h^3} - \frac{1}{h_\infty^3}\right) = \frac{2\gamma\cos\theta}{d}$$

Let $h = h_\infty + \delta h$ where $\delta h \ll h_\infty$ so that $\underbrace{\frac{1}{h^3} = \frac{1}{h_\infty^3\left(1+\frac{\delta h}{h_\infty}\right)^3} = \frac{1}{h_\infty^3}\left(1 - 3\frac{\delta h}{h_\infty}\right)}_{\text{Linearization}}$

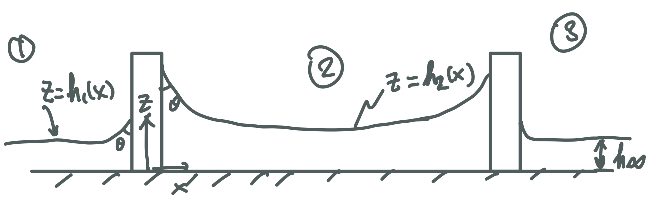
$$\rightarrow \frac{A_{SLV}}{6\pi}\left(-\frac{3\,\delta h}{h_\infty^4}\right) = \frac{2\gamma\cos\theta}{d}, \quad \text{i.e.,} \quad \delta h = \frac{4\pi\gamma\cos\theta\, h_\infty^4}{(-A_{SLV})\, d}$$

In this case, the interation takes the form

$$\frac{F}{W} = \delta h \left(\frac{A_{SLV}}{6\pi h_\infty^3}\right) = -\frac{2}{3}\frac{\gamma h_\infty \cos\theta}{d} \quad \text{(Always attractive!)}$$

↑ Unlike gravity, p is a constant here.

But what is meant by "small d" ? – There should be a length-scale below like l_c



For both ① and ②, we have

$$P_{atm} + \gamma \cdot \nabla \underset{\sim}{n} + \frac{A_{SLV}}{6\pi h^3} = P_{atm} + \frac{A_{SLV}}{6\pi h_\infty^3}$$

or

$$h_{xx} - \frac{A_{SLV}}{6\pi\gamma}\frac{1}{h^3} = -\frac{A_{SLV}}{6\pi h_\infty^3}$$

Consider $h \sim h_\infty$, a natural lateral length is

$$l_{vdw} = \left(\frac{2\pi\gamma h_\infty^4}{|A_{SLV}|}\right)^{1/2} \sim 100\ \mu m$$

$\gamma \sim 0.1$ N/m

$A \sim 10^{-20}$ J

$h_\infty \sim 100$ nm ? why

Also note that gravity is neglected, implying

$$l_{vdw} << l_c \rightarrow \frac{6\pi\gamma h_\infty^4}{|A_{SLV}|} << \frac{\gamma}{\rho g} \rightarrow h_\infty << \left(\frac{A_{SLV}}{6\pi\rho g}\right)^{1/4} \sim \left(\frac{10^{-20} J}{6\pi \times 10^4 \frac{N}{m^3}}\right)^{1/4} \simeq 500\,nm$$

This may appear more natural in a linearized version $h = h_\infty + \delta h$

$$\delta h_{xx} - \frac{A_{SLV}}{6\pi\gamma h_\infty^3}(1 - 3\delta h) = \frac{A_{SLV}}{6\pi\gamma h_\infty^3}$$

$$\Rightarrow \delta h_{xx} - \underbrace{\frac{A_{SLV}}{2\pi\gamma h_\infty^3}}_{1/l_{vdw}^2}\delta h = 0$$

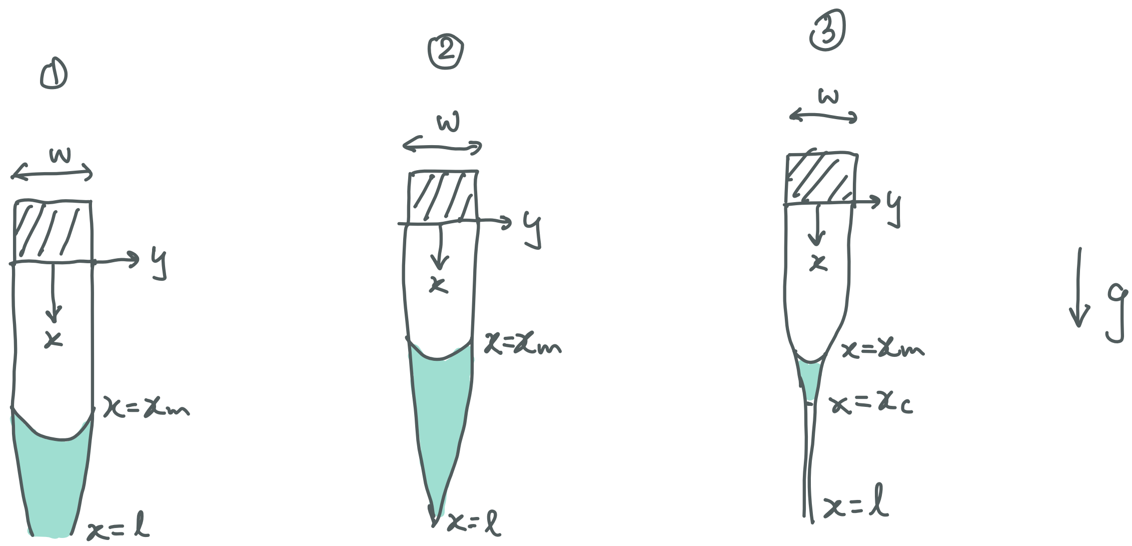
It may be a course project to study linearized and fully nonlineary version

of lateral immersion forces for $\theta_1 \neq \theta_2$.

Elastocapillarity

• Elastocapillary rise

Slender structures are often very bendable. The question is how the elastic and surface energies interplay. We then consider the following problem (Kim & Mahadevan JFM 2006) – Capillary rise between elastic sheets



More relevant examples: wetted paintbrush/hairs, closure of airways within the lung, and stiction of MEMS.

Aside: As $t \ll l$ (geometrically slender) and $w \ll l$ (geometrically linear), it is possible to use $\varepsilon = t/l$ as the small paremeter, reducing Navier Cauchy equations to linear plate equation / Eular-Bernoulli beam equation:

$$\nabla^4 y(\underset{\sim}{x}) = p(\underset{\sim}{x})$$

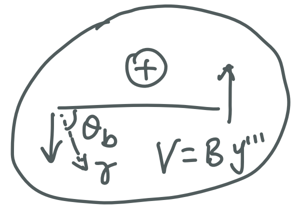
• Case (1)

No pressure in the "dry" part : $y'''' = 0$, $0 < x < x_m$

In the "wet" part, $\underset{\substack{\uparrow \\ \text{Bending} \\ \text{stiffness}}}{B} y'''' = -\gamma \underset{\substack{\uparrow \\ \text{Curvature of the meniscus at the bottom } (x=l)}}{\cancel{\kappa_b}}^{\nearrow 0 \text{ in experiments}} - \rho g (l - x)$

Need 9 boundary conditions / matching conditions to solve the 2 fourth-order ODE + the unknown x_m.

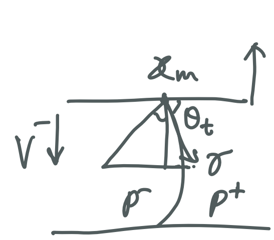
- At $x=0$, $y(0) = \frac{1}{2} W$, $y'(0) = 0$



- At $x = l$, $y''(l) = 0$, $B y'''(l) = \gamma \sin\theta_b \simeq \gamma$

- At $x = x_m$, $[y(x_m)] = [y'(x_m)] = [y''(x_m)] = 0$

$$[f(x_m)] = f(x_m^+) - f(x_m^-)$$



$$B[y'''(x_m)] = \gamma \sin\theta_t$$

$$p^+ - p^- = -\rho g (l - x_m) = -\gamma \cos\theta_t / y(x_m)$$

Use l to scale the coordinate and w to scale the sheet deflection

$$X = x/l \ , \ Y = y/w \ , \ l_c = \left(\frac{\gamma}{\rho g}\right)^{1/2} , \ l_{ec} = \left(\frac{B}{\gamma}\right)^{1/2}$$

$$\Rightarrow Y'''' = \begin{cases} 0 \ , & 0 < X < X_m \\ \dfrac{l^5}{l_c^2 l_{ec}^2 w} (X - 1) \ , & X_m < X < 1 \end{cases}$$

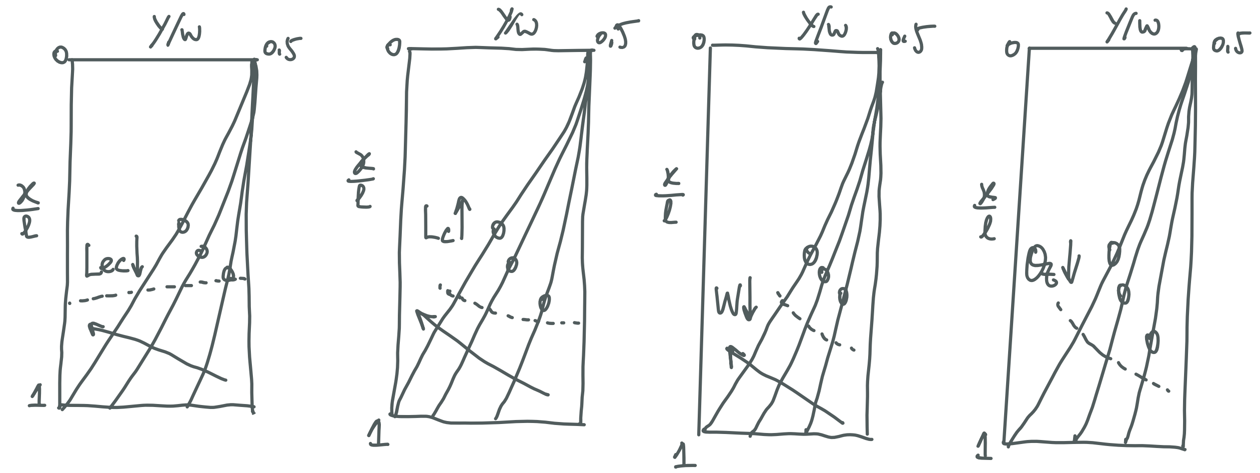
subject to

$$Y(0)=\frac{1}{2}, \quad Y'(0)=0, \quad Y''(1)=0, \quad Y'''(1)=\frac{\ell^3}{w\,\ell_{ec}^2}$$

$$[Y(X_m)]=[Y'(X_m)]=[Y''(X_m)]=0$$

$$[Y'''(X_m)]=\frac{\ell^3}{w\,\ell_{ec}^2}\sin\theta_t, \quad Y(X_m)=\frac{\ell_c^2}{\ell w}\cos\theta_b\,(1-X_m)^{-1}$$

Key parameters: L_{ec}, L_c, $W=w/\ell$ and θ_t



..... Capillary rise between rigid sheets.

- Case ②

The same except for $By'''(\ell)=\gamma \rightarrow y(\ell)=0$

- Case ③

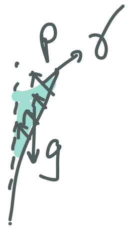
$$p(x)=\begin{cases} 0, & 0<x<x_m \\ \rho g(x-x_c), & x_m<x<x_c \end{cases}$$

BCs and matching conditions are the same at $x=0$ & $x=x_m$ (7 conditions)

$y(x_c)=y'(x_c)=0$

$y''(x_c)=0 \rightarrow$ To solve x_c

• Vertical force balance



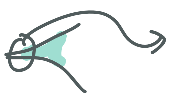
$$\int_{x_m}^{x_c} \rho g y \, dx = \gamma \cos\theta_t - \underbrace{\int_{x_m}^{x_c} B y'''' y' dx}$$

$$\int_{x_m}^{x_c} B y'''' y' dx = \int_{x_m}^{x_c} \rho g (x - x_c) y' dx$$

$$= \rho g (x - x_c) y \Big|_{x_m}^{x_c} - \int_{x_m}^{x_c} \rho g y \, dx$$

$$= -\rho g (x_m - x_c) y(x_m) - \int_{x_m}^{x_c} \rho g y \, dx$$

$$\Rightarrow \rho g (x_m - x_c) = \frac{-\gamma \cos\theta_t}{y(x_m)} \quad (\checkmark \text{ self-constant})$$

•  $2 \times \frac{1}{2} B y''^2(x_c) = 2\gamma_{sl} - \gamma_{ss} = -S = \Gamma - 2\gamma\cos\theta_t$

$\Gamma = 2\gamma_{sv} - \gamma_{ss}$

Interface spreading parameter

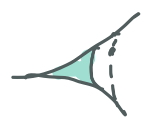
$S > 0$, $\gamma_{ss} > 2\gamma_{sl}$, $\rightarrow$ There will be a thin layer of liquid.

(Occurs for small Γ, large γ, small θ_t $\rightarrow$ $y''(x_c) = 0$)

$S < 0$, $\gamma_{ss} < 2\gamma_{sl}$ $\rightarrow$ There will be a jump in curvature

(Occurs for large Γ, small γ, large θ_t $\rightarrow$ $y''(x_c) = (-S/B)^{\frac{1}{2}}$).

• If $S > 0$, $y''(x_c) = 0$. As $x_0 = x_c - x_m \to 0$, whether the problem

will decay to $y'''' = 0$, $0 < x < x_c$, subject to 

$$y(0) = \tfrac{1}{2} w, \quad y'(0) = 0, \quad y(x_c) = 0, \quad y'(x_c) = 0, \quad y'' = \left[\frac{2(\gamma_{sv} - \gamma_{sl})}{B}\right]^{\frac{1}{2}}$$

$$?? = \left(\frac{2\gamma\cos\theta_t}{B}\right)^{\frac{1}{2}}$$