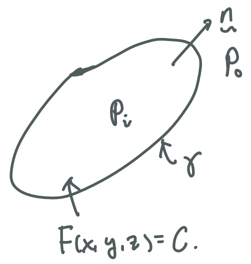
Fluid Statics

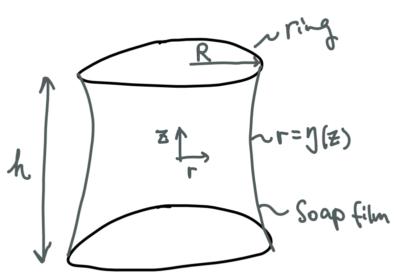
In last lecture, we discussed the pressure jump across an interface, i.e., Laplace's theorem



$$\Delta P = P_i - P_o = \gamma \nabla\cdot \underset{\sim}{n}, \quad \text{where } \underset{\sim}{n} = \frac{\nabla F}{|\nabla F|}$$

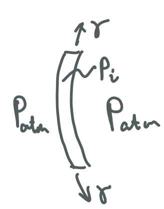
In this lecture, we discuss a number of examples, some of which can lead to the measurement of surface tension of liquids.

· Minimal surfaces (Plateau's problem raised by Lagrange in 1760)



Determine $\eta(z)$?

The pressure jump has to be zero:



$$\underset{\substack{\uparrow \\ \text{Left}}}{P_{atm}} = \underbrace{P_{atm} + \gamma \nabla\cdot\underset{\sim}{n}}_{P_i} + \gamma \nabla\cdot \underset{\sim}{n}$$

This problem becomes solving minimal surfaces with zero mean curvature.

$$F(r,z) = r - \eta(z) = 0$$

$$\underset{\sim}{n} = \frac{\nabla F}{|\nabla F|} = \frac{\underset{\sim}{e_r} - \eta_z \underset{\sim}{e_z}}{(1+\eta_z^2)^{1/2}}$$

$$K = \nabla \cdot \underset{\sim}{n} = \frac{1}{r}\frac{\partial}{\partial r}(r \cdot n_r) + \frac{\partial}{\partial z}(n_z)$$

$$= \frac{1}{\eta(1+\eta_z^2)^{1/2}} - \left[\frac{\eta_{zz}}{(1+\eta_z^2)^{1/2}} - \frac{\eta_z^2 \eta_{zz}}{(1+\eta_z^2)^{3/2}}\right]$$

$$= \frac{1}{(1+\eta_z^2)^{1/2}}\left(\frac{1}{\eta} - \frac{\eta_{zz}}{1+\eta_z^2}\right) = 0$$

Aside

- $(\eta_z^2)' = 2\eta_z \eta_{zz}$
- $\left((1+\eta_z^2)^{-1/2}\right)' = -(1+\eta_z^2)^{-3/2}\eta_z \eta_{zz}$

Typical tricks to reduce the order of ODE for surface tension problems

We obtain $\boxed{\eta \eta_{zz} = 1 + \eta_z^2}$ for $-\frac{h}{2} \leq z \leq \frac{h}{2}$

A trick that is useful here is that $(1+\eta_z^2)' = 2\eta_z \eta_{zz}$

$$\rightarrow \frac{\eta(1+\eta_z^2)'}{2\eta_z} = 1 + \eta_z^2 \quad \text{or} \quad \frac{\eta_z}{\eta} = \frac{(1+\eta_z^2)'}{2(1+\eta_z^2)}$$

We then have $\ln \eta = \frac{1}{2}\ln(1+\eta_z^2) + c$, which can be rewritten as

$$\eta = C(1+\eta_z^2)^{1/2} \rightarrow \eta_z = \frac{\sqrt{\eta^2 - c^2}}{c} = \frac{d\eta}{dz} \rightarrow dz = \frac{c}{\sqrt{\eta^2 - c^2}}\, d\eta$$

$$\rightarrow \left[C \cosh^{-1}\left(\frac{\eta}{c}\right)\right]'$$

The solution reads $z - z_0 = c \cosh^{-1}\left(\frac{\eta}{c}\right)$, or

$$\eta = c \cosh\left(\frac{z - z_0}{c}\right).$$

Use boundary conditions to determine integration constants C & z_0.

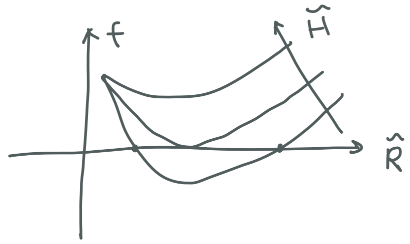
Symmetry at $z=0$, $y_z=0 \rightarrow z_0=0$

Fixed edge at $z=\frac{h}{2}$, $y(\frac{h}{2})=R \rightarrow R=C\cosh\left(\frac{h}{2C}\right)$

$$\boxed{\text{Therefore, the solution is } y=C\cosh\left(\frac{z}{C}\right), \text{ where } C \text{ satisfies } R=C\cosh\left(\frac{h}{2C}\right).}$$

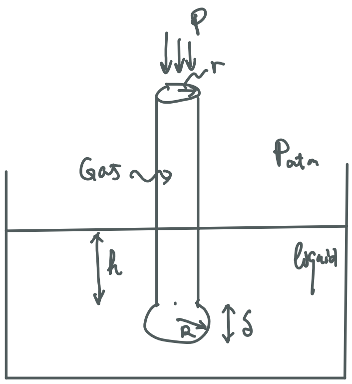
Catenary curve.

Let $\hat{R}=R/C$, $\hat{H}=h/R$. Then seek solution $f(\hat{R},\hat{H})=\cosh\left(\frac{1}{2}\hat{H}\hat{R}\right)-\hat{R}=0$

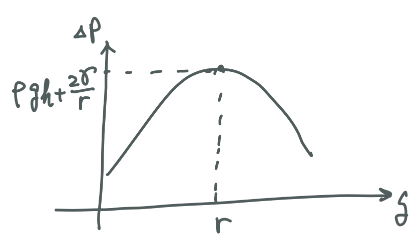


- This equation has two solutions for C when h is not too greater than R: one is for minimal surface and the other is for maximal surface.
- When $\hat{H}=1.33$, i.e., $h=1.33R$, the two solutions are identical. For $h>1.33R$, No solutions (soap film bursts).

- **Maximal pressure of a bubble** (E. Schrödinge one of pioneers)



$$P=P_{atm}+\rho g h+\frac{2\gamma}{R}$$

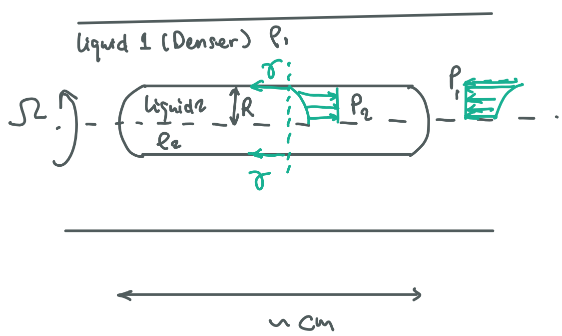


$h \gg r \rightarrow$ bubble is approximately spherical

Precise and robust – works even at very high temperatures, allowing experiments with metals and molten glass. And bubble breaking will "refresh" the interface.

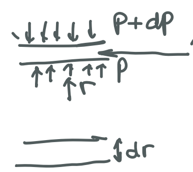
↑ Preventing from potential contaminants

· Spinnig drops



Force balance method.

$$\int_0^R 2\pi P_2 \, r \, dr = \int_0^R 2\pi P_1 \, r \, dr + 2\pi R \gamma$$



$\uparrow F_c \sim$ Centrifugal force

$$\rho dV \frac{u_\theta^2}{r} = dp \cdot dA \rightarrow \frac{dp}{dr} = \rho \omega^2 r \qquad u_\theta = \omega r$$

We then have $P_i = C_i + \frac{1}{2}\rho_i \omega^2 r^2$ with C_1, C_2 two unknown integration contants.

① At $r=R$, we know $P_2 - P_1 = \frac{\gamma}{R}$

$$P_1(R) = C_1 + \frac{1}{2}\rho_1 \omega^2 R^2$$
$$P_2(R) = C_2 + \frac{1}{2}\rho_2 \omega^2 R^2 \quad \rightarrow \quad C_2 - C_1 = \frac{1}{2}(\rho_2 - \rho_1)\omega^2 R^2 + \frac{\gamma}{R}$$

② Horizontal force balance

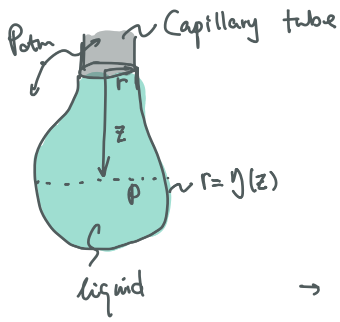
$$\pi R^2 \cdot C_2 + \frac{\pi}{4}\rho_2 \omega^2 R^4 = \pi R^2 \cdot C_1 + \frac{\pi}{4}\rho_1 \omega^2 R^4 + 2\pi R \gamma$$

$$\rightarrow C_2 - C_1 = \frac{1}{4}(\rho_1 - \rho_2)\omega^2 R^2 + \frac{2\gamma}{R}$$

$$\Rightarrow \boxed{\gamma = \frac{1}{4}(\rho_1 - \rho_2)\omega^2 R^3}$$

☆ A great advantage: It Does not involve contact with a solid.

• Pendant drops



$$P(z) = P_{atm} + \rho g z$$

$$P(z) = P_{atm} + \gamma \cdot \nabla \underline{n}$$

$$\rightarrow \gamma \cdot \nabla \underline{n} - \rho g z = 0 \quad , \text{ where } \nabla \underline{n} = - \frac{y_{zz}}{(1+y_z^2)^{3/2}} + \frac{1}{y(1+y_z^2)^{1/2}}$$

Strategy: Solving ODE numerically and using γ as a fitting parameter to match experimental results. The error is within %.

Note: When the drop's weight exceeds the capillary force acting on the edge of the tube $2\pi R\gamma$, drop drops!

$\sim 60\% \rightarrow 40\%$ remainder on the tube

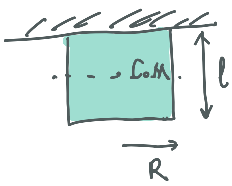
$$2\pi R \gamma = \alpha \frac{4}{3}\pi R_g^3 \times \rho g$$

$$\rightarrow \quad R_g = \left(\frac{3}{2\alpha} R\, l_c^2 \right)^{1/3} \quad \sim \text{ millimeter scale.}$$

• Stability of pendant drop

Note that previous discussions are all about $\delta F = 0$. This gives extrema but does not specify whether it is a minimum or a maximum. In the problem of soap film between two rings, we have realized that both could be solved as solutions but one of the two appears to be a maximum – which is unstable. Similar scenarios occur also in the pendant drop problem.

First, consider a simplified model problem – a cylinder



$$F_V = \underbrace{-\rho g V \times \tfrac{1}{2} l}_{\text{Gravitational E.}} + \overbrace{\gamma(\pi R^2 + 2\pi R l) + (\gamma_{SL} - \gamma_{SV})\pi R^2}^{\text{Surface energy}}$$

R, l are not independent – They satisfy $\pi R^2 l = V$

Let us rewrite $F_V(R, l)$ as $F(l, V)$

$$F = -\tfrac{1}{2}\rho g V l + \underbrace{(\gamma + \gamma_{SL} - \gamma_{SV})}_{\gamma(1-\cos\theta)}\frac{V}{l} + \gamma(4\pi V l)^{1/2}$$

May also use l_c as the characteristic length.

Without loss, Let us try $\theta = \pi/2$ and rescale F by $\gamma V^{2/3}$ and l by $V^{1/3}$, say

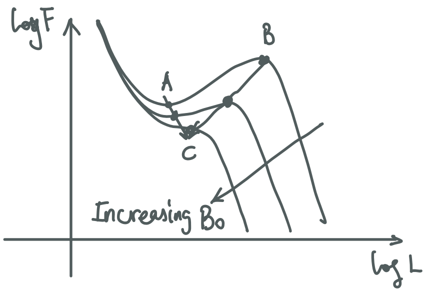
$$\mathrm{F} = F/\gamma V^{2/3}, \quad L = l/V^{1/3}, \quad l_c = (\gamma/\rho g)^{1/2}, \quad B_o = \left(\frac{V^{1/3}}{l_c}\right)^2$$

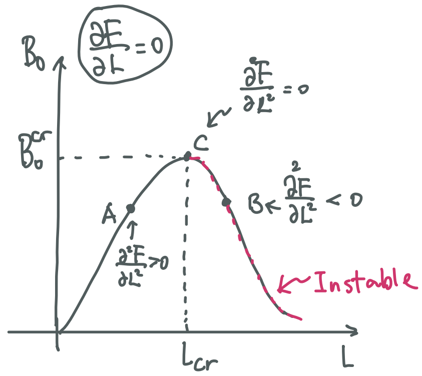
We now have

$$\mathrm{F} = -\frac{1}{2}\overbrace{\left(\frac{V^{1/3}}{l_c}\right)^2}^{B_o} L + \frac{1}{L} + (4\pi)^{1/2} L^{1/2}$$

Indeed, $\left.\frac{\partial^2 \mathrm{F}}{\partial L^2}\right|_{L_{cr}} = 0$ gives $\frac{2}{L_{cr}^3} - \frac{\pi^{1/2}}{2L_{cr}^{3/2}} = 0 \to L_{cr} \equiv \left(\frac{16}{\pi}\right)^{1/3}$ (*)

At this moment, $\left.\frac{\partial \mathrm{F}}{\partial L}\right|_{L_{cr}} = 0$ gives $-\frac{1}{2}B_o^{cr} - \frac{1}{L_{cr}^2} + \left(\frac{\pi}{L_{cr}}\right)^{1/2} = 0 \to B_o^{cr} = \frac{3}{2}\left(\frac{\pi}{2}\right)^{2/3}$

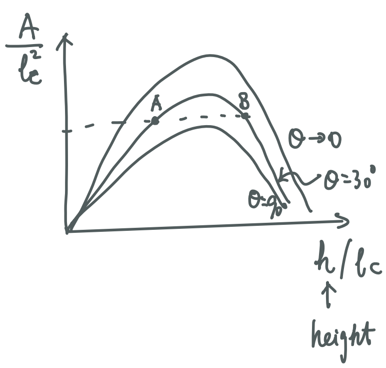


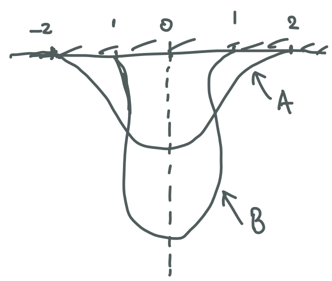


$$\Rightarrow \boxed{\begin{aligned} &V_{cr} = (B_0^{cr})^{3/2}\, \ell_c^3 \simeq 2.89 \left(\frac{\gamma}{\rho g}\right)^{\frac{3}{2}} && \text{Critical volume} \\ &\ell_{cr} = V_{cr}^{1/3}\, \ell_c \simeq 1.42 \left(\frac{\gamma}{\rho g}\right)^{1/2} && \text{Critical height} \\ &R_{cr} = \left(V_{cr}/\pi \ell_{cr}\right)^{1/2} \simeq 0.80 \left(\frac{\gamma}{\rho g}\right)^{1/2}. && \text{Critical radius} \end{aligned}}$$

Formal analysis was provided by E. Pitts, JFM (1973) & (1974)

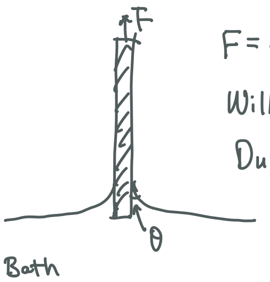
- In JFM1973 paper, a 2D case was analyzed





- In JFM 1974 paper, an axisymmetric case was considered.

· Force measurements

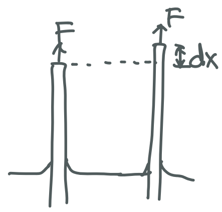


$F = l \times \gamma \cos\theta$

Wilhelmy plate

Du Noüy ring

⋮



Ex: Fiber

$F\,dx = 2\pi R\,dx\,(\gamma_{sv} - \gamma_{sl})$

$= 2\pi R\,\gamma\cos\theta$

$\rightarrow l_{fiber} = 2\pi R$

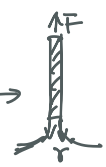
How to eliminate θ so that $F = l \times \gamma$

(1) Using a solid with high surface energy – wettable by all usual liquids ($\theta = 0$)

Contaminants that are spontaneously absorbed on the surface would lower the γ_{solid}.

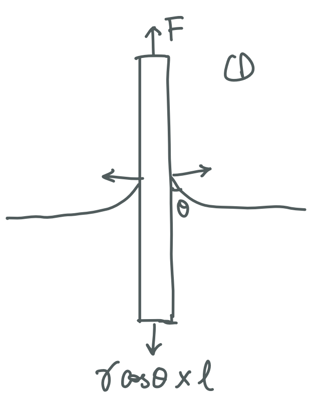
- Platinum – surface can regenerate by a flame.

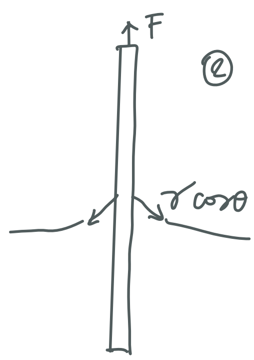
(2) Quasi-static loading. $F_{max} = l \times \gamma \rightarrow$

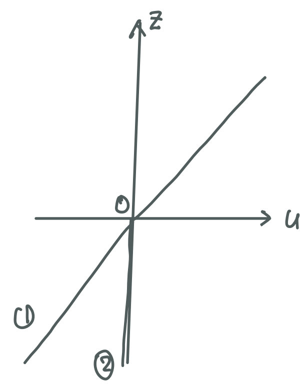


· Stresses in slender, soft solids

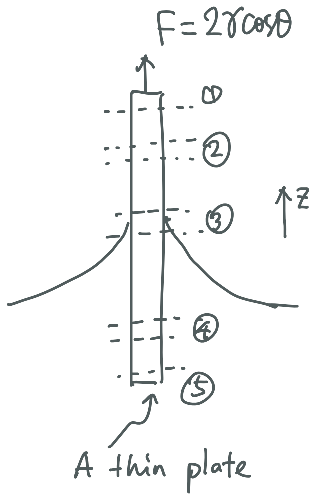
What is the consequence of this $\gamma\cos\theta$? – Need some thought



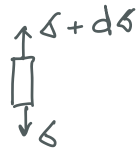




Need to consider the boundary value problem, which can be derived via variational analysis. But here I decided to use the method of force/stress balance.



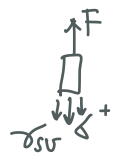
Governing equations:

② : 

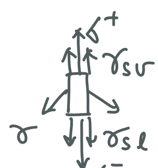
$$\frac{d\sigma}{dz} = 0 \rightarrow \sigma \equiv \sigma^+ , \quad z > 0$$

④ $\sigma \equiv \sigma^- , \quad z < 0$

Boundary Conditions:

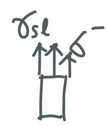
① 

$$\sigma^+ t = F - 2\gamma_{sv} \rightarrow \sigma^+ = \frac{2\gamma\cos\theta - 2\gamma_{sv}}{t}$$

③ 

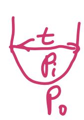
$$\sigma^- t = \sigma^+ t + \underbrace{2\gamma_{sv} - 2\gamma\cos\theta - 2\gamma_{sl}}_{=0}$$

$$\rightarrow \sigma^- = \sigma^+$$

⑤ 

$$\sigma^- = \frac{-2\gamma_{sl}}{t} \quad (= \sigma^+ \checkmark)$$

Does not depend on shape, say

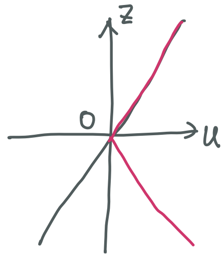


$$\sigma = -(P_i - P_o) = -\frac{\gamma_{sl}}{t/2}$$

This means there's no stress jump if all γ's are constant!!!

Since measures are made relative to $\sigma_0 = -2\gamma_{sv}$, we expect that

$$\sigma^{+}_{Exp} = \sigma^{-}_{Exp} = \frac{2\gamma\cos\theta}{t}$$ or



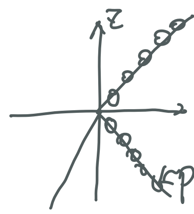
$$\boxed{\frac{du}{dz} = \epsilon_z = \frac{2\gamma\cos\theta}{Et}}$$

However, experiments by Marchand et al. PRL (2012) on small fibers showed the red curve, which needs further verification via plate experiments!

$$\sigma_{zz} = \begin{cases} \dfrac{2\gamma\cos\theta}{R} - \dfrac{2\gamma_{sv}}{R}, & z>0 \\ -\dfrac{2\gamma_{sl}}{R}, & z<0 \end{cases} \qquad \sigma_{\theta\theta} = \sigma_{rr} = \begin{cases} -\dfrac{\gamma_{sv}}{R}, & z>0 \\ -\dfrac{\gamma_{sl}}{R}, & z<0 \end{cases}$$

$$\Rightarrow \sigma_{zz}^{Exp} = \frac{2\gamma\cos\theta}{R}, \quad \sigma_{\theta\theta}^{Exp} = \sigma_{rr}^{Exp} = \begin{cases} 0, & z>0 \\ \dfrac{\gamma\cos\theta}{R}, & z<0 \end{cases}$$

$$\Rightarrow \epsilon_{zz} = \frac{\sigma_{zz} - \overset{\approx 1/2}{\nu}(\sigma_{rr} + \sigma_{\theta\theta})}{E} = \begin{cases} \dfrac{2\gamma\cos\theta}{R} \\ \dfrac{\gamma\cos\theta}{R} \end{cases}$$



PRL (2012). why?

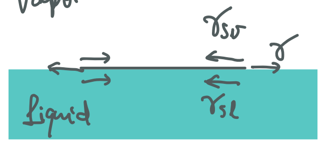
• Stresses in soft thin sheets

A recent work by Kumar et al. Nat. Mater. (2020) did use a thin plate/sheet.

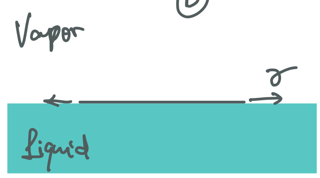
However, the stress/strain field are not directly measured.

The question posed by this work is which boundary condition to use.

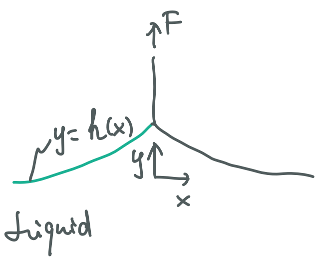
A



B



The experimental set-up goes as following



$$F = \gamma + \gamma_{sv} - \gamma_{sl}$$
$$= \gamma(1+\cos\theta)$$

- For liquid surface

$$\gamma \nabla\cdot n + \rho g h = 0, \quad \nabla\cdot n = -\frac{h_{xx}}{(1+h_x)^{3/2}}$$

Subject to $h(\infty)=0$, $h(0)=\delta$ (prescribed)

- For thin sheet on liquid surface

$$\sigma \nabla\cdot n + \rho g h = 0, \quad h(\infty)=0, \quad h(0)=\delta.$$

Then it is found that the two profiles are identical, regardless of the liquid used.

$$\Rightarrow \sigma = \gamma, \quad \text{i.e.} \ \sigma \longleftrightarrow \gamma \ \text{(B)}$$

However, if we go through the variation, the true governing equation should be

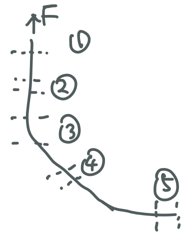
$$(\sigma_v + \gamma_{sl} + \gamma_{sv})\nabla\cdot n + \rho g h = 0$$

subject to (A) $\sigma_v + \gamma_{sl} + \gamma_{sv} = \gamma$. Let $\sigma = \sigma_v + \gamma_{sl} + \gamma_{sv}$, leading to the same results. → Both (A) & (B) can be used, but should be cautious about the reference state! The question of (A) or (B) is not answered!

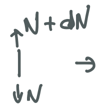
A possible way to address this problem could be given by examining the $\sigma(x)$!

in the sheet.

(A) :

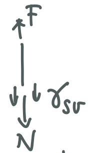


(2) , (4)



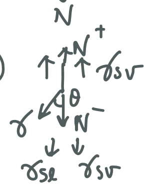
$\rightarrow N = C$

(1)



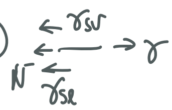
$N^+ = F - 2\gamma_{sv} = \gamma - \gamma_{sv} - \gamma_{sl}$

(3)



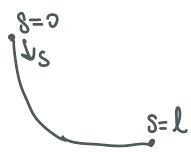
$N^- = N^+$

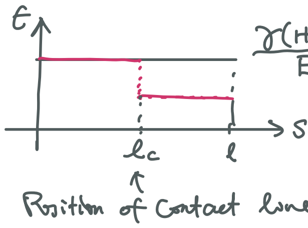
(5)



$N^- = \gamma - \gamma_{sv} - \gamma_{sl}$ ✓.

$\Rightarrow$





$\frac{\gamma(1+\cos\theta)}{E}$ (Relative to $-2\gamma_{sv}$)

(B) would give a jump in ϵ, shown as the red curve.