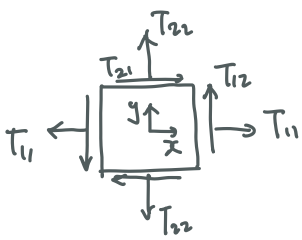
Stress balance on a free surface

• Stress jump conditions

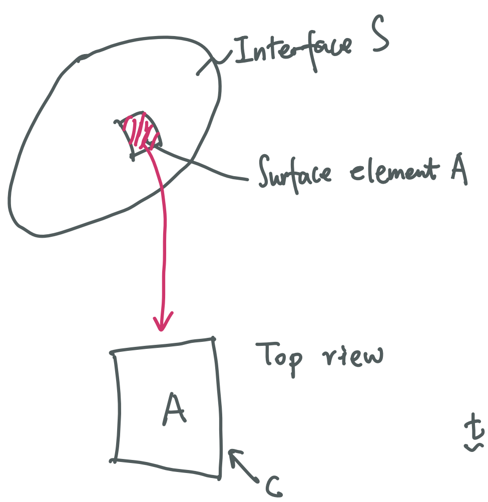
Usually, via variational analysis, we are able to derive the governing equation(s) for the system and appropriate bcs to solve such equations.

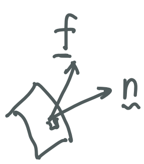
However, we have discussed the key bc(s) may be directly "illustrated" by line force balance at "sort-of" triple phase contact line. We indeed can also obtain governing equation in a similar manner - stress balance!

$$\text{Stress tensor} \quad \underset{\approx}{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \text{FORCE/AREA}.$$



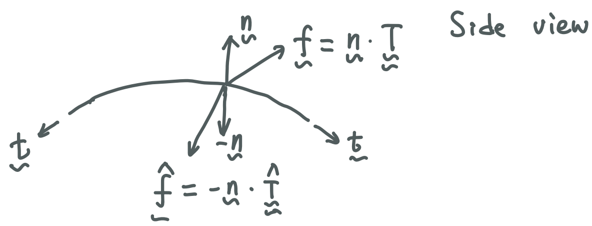
Balance of moments $\rightarrow T_{ij} = T_{ji}$





$$\underset{\sim}{f} = \underset{\sim}{n} \cdot \underset{\approx}{T} \qquad \text{(Balance of forces)}$$

$$f_i = T_{ij} n_j$$



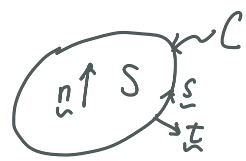
In equilibrium, $\sum \mathbf{F} = m\mathbf{a} = 0$

$$\left\{\begin{matrix}\text{Force Acting on} \\ \text{the top surface}\end{matrix}\right\} + \left\{\begin{matrix}\text{Force Acting on} \\ \text{the bottom surface}\end{matrix}\right\} + \left\{\begin{matrix}\text{Surface tension} \\ \text{Acting on Perimeter}\end{matrix}\right\} = 0$$

$$\int_A \mathbf{n} \cdot \mathbf{T}\, dA + \int_A -\mathbf{n} \cdot \hat{\mathbf{T}}\, dA + \oint_C \gamma \mathbf{t}\, dl = 0$$

• **Stokes' theorem:** "The line integral of a vector field over a loop is equal to the flux of it curl through the enclosed surface."

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \int_S \mathbf{n} \cdot (\nabla \times \mathbf{F})\, ds = \int_C \mathbf{F} \cdot \mathbf{s}\, dl$$



Let $\mathbf{F} = \mathbf{f} \times \mathbf{b}$, where b is an arbitrary constant vector. We thus have

$$\oint_C (\mathbf{f} \times \mathbf{b}) \cdot \mathbf{s}\, dl = \int_S \mathbf{n} \cdot (\nabla \times (\mathbf{f} \times \mathbf{b}))\, ds$$

Now we use standard vector identities to see:

- $(\mathbf{f} \times \mathbf{b}) \cdot \mathbf{s} = -\mathbf{b} \cdot (\mathbf{f} \times \mathbf{s})$
- $\nabla \times (\mathbf{f} \times \mathbf{b}) = \mathbf{f}\, \cancel{(\nabla \cdot \mathbf{b})}^{\,0} - \mathbf{b}(\nabla \cdot \mathbf{f}) + \mathbf{b} \cdot (\nabla \mathbf{f}) - \mathbf{f} \cdot \cancel{(\nabla \mathbf{b})}^{\,0}$

$\mathbf{b}$ is constant vector

We then have

$$-\mathbf{b} \cdot \oint_C (\mathbf{f} \times \mathbf{s})\, dl = -\mathbf{b} \int_S \left[\mathbf{n}(\nabla \cdot \mathbf{f}) - (\nabla \mathbf{f}) \cdot \mathbf{n}\right] ds$$

$\mathbf{b}$ is arbitrary

$$\Rightarrow \quad \oint_C (\mathbf{f} \times \mathbf{s})\, dl = \int_S \left[\mathbf{n}(\nabla \cdot \mathbf{f}) - (\nabla \mathbf{f}) \cdot \mathbf{n}\right] ds$$

$$\nabla \mathbf{f} = \frac{\partial f_j}{\partial x_i} \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{i} \otimes \mathbf{j}$$

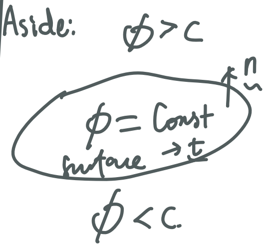
$$\mathbf{n} \cdot (\mathbf{b} \cdot (\nabla \mathbf{f})) = (\mathbf{b} \cdot \mathbf{i})(\mathbf{n} \cdot \mathbf{j}) = \mathbf{b} \cdot (\mathbf{i} \otimes \mathbf{j} \cdot \mathbf{n})$$

We may choose $\underset{\sim}{f} = \gamma \underset{\sim}{n}$, and recall $\underset{\sim}{n} \times \underset{\sim}{s} = -\underset{\sim}{t}$. One thus obtain

$$-\oint_C \gamma \underset{\sim}{t}\, dl = \int_S \left[\underset{\sim}{n}\, \underset{\sim}{\nabla}\cdot(\gamma \underset{\sim}{n}) - \underset{\sim}{\nabla}(\gamma \underset{\sim}{n})\cdot n \right] dS$$

$$= \int_S \left[\underset{\sim}{n} \underbrace{(\underset{\sim}{\nabla}\gamma \cdot \underset{\sim}{n})}_{=\frac{\partial \gamma}{\partial n} = 0} + \underset{\sim}{n}\, \gamma (\underset{\sim}{\nabla}\cdot\underset{\sim}{n}) - \underbrace{(\underset{\sim}{\nabla}\gamma \otimes \underset{\sim}{n})\cdot\underset{\sim}{n}}_{=\underset{\sim}{\nabla}\gamma} - \gamma \underbrace{(\underset{\sim}{\nabla}\underset{\sim}{n})\cdot\underset{\sim}{n}}_{=\frac{1}{2}\nabla(n\cdot n) = \frac{1}{2}\nabla 1 = 0} \right] dS$$

Aside:



$$\underset{\sim}{t}\cdot\underset{\sim}{\nabla}\phi = \frac{dx_1}{ds}\frac{\partial\phi}{\partial x_1} + \frac{dx_2}{ds}\frac{\partial\phi}{\partial x_2} + \frac{dx_3}{ds}\frac{\partial\phi}{\partial x_3} = \frac{\partial\phi}{\partial s} = 0 \quad \text{for } \underset{\sim}{t} \text{ on surface}$$

$$\to \underset{\sim}{t} \perp \underset{\sim}{\nabla}\phi \quad \text{or} \quad \underset{\sim}{n} \parallel \underset{\sim}{\nabla}\phi$$

$$\Rightarrow \boxed{\oint_C \gamma \underset{\sim}{t}\, dl = \int_A \underset{\sim}{\nabla}\gamma\, dA - \int_A \gamma \underset{\sim}{n}\,(\underset{\sim}{\nabla}\cdot\underset{\sim}{n})\, dA}$$

$$\nabla = \frac{\partial}{\partial x}\underset{\sim}{e_x} + \frac{\partial}{\partial y}\underset{\sim}{e_y} + \frac{\partial}{\partial z}\underset{\sim}{e_z}, \quad \underset{\sim}{\nabla}\cdot\underset{\sim}{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

We then have $\int_A \left[\underset{\sim}{n}\cdot(\underset{\approx}{T} - \underset{\approx}{\hat{T}}) + \underset{\sim}{\nabla}\gamma - \gamma \underset{\sim}{n}\,(\nabla\cdot\underset{\sim}{n}) \right] dA = 0$

This is true for arbitrary A $\to$ the integrand must vanish. One thus obtains

the <u>interfacial stress balance equation</u>:

$$\boxed{\underset{\sim}{n}\cdot(\underset{\approx}{T} - \underset{\approx}{\hat{T}}) + \nabla\gamma - \gamma \underset{\sim}{n}\,(\nabla\cdot\underset{\sim}{n}) = 0}$$

Tangential stress associated with gradients in γ

Normal curvature force per unit area associated with local curvature of interface $\nabla\cdot\underset{\sim}{n}$

• Normal stress balance

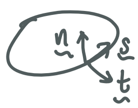
$\nabla\gamma$ must be tangent to the surface

$$\mathbf{n}\cdot(\mathbf{T}-\hat{\mathbf{T}})\cdot\mathbf{n} + \cancelto{0}{\nabla\gamma\cdot\mathbf{n}} - \gamma\,\cancelto{1}{(\mathbf{n}\cdot\mathbf{n})}\,(\nabla\cdot\mathbf{n}) = 0$$

$$\rightarrow \boxed{\mathbf{n}\cdot(\mathbf{T}-\hat{\mathbf{T}})\cdot\mathbf{n} = \gamma\,\nabla\cdot\mathbf{n}}$$

The jump in normal stress ("pressure") across the interface is balanced by the curvature pressure.

• Shear stress balance



$\mathbf{m}$ is any combination of $\mathbf{s}$ and $\mathbf{t}$

$$\mathbf{n}\cdot(\mathbf{T}-\hat{\mathbf{T}})\cdot\mathbf{m} + \nabla\gamma\cdot\mathbf{m} - \gamma\,\cancelto{0}{(\mathbf{n}\cdot\mathbf{m})}\,(\nabla\cdot\mathbf{n}) = 0$$

$$\rightarrow \boxed{\mathbf{n}\cdot(\mathbf{T}-\hat{\mathbf{T}})\cdot\mathbf{m} + \nabla\gamma\cdot\mathbf{m} = 0}$$

The jump in tangential stress (Marangoni stress) across the interface is caused by the gradients in γ — may result from gradients in temperature T or chemical composition c at the interface since in general $\gamma = \gamma(T, c)$.

Non-zero $\nabla\gamma \rightarrow$ flow !

Back to the stress tensor, rewritten as

$$\mathbf{T} = -p\,\mathbf{I} + \boldsymbol{\tau}$$

Neg. sign due to pressure

Deviatoric stress

Consider fluid static (no flow), $\boldsymbol{\tau} = \hat{\boldsymbol{\tau}} = 0$, $\nabla\gamma = 0$. Plugging stress tensor into normal stress balance equation,

$$\mathbf{n}\cdot(\hat{p}-p)\,\mathbf{I}\cdot\mathbf{n} - \gamma\,\nabla\cdot\mathbf{n} = 0 \;\rightarrow\; \boxed{\Delta p = \hat{p} - p = \gamma\,\nabla\cdot\mathbf{n}}$$

Laplace pressure

Take gradient on both side, we have

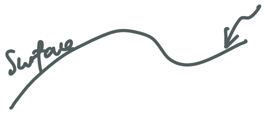
$$\nabla(\hat{p}-p) = \nabla(\gamma\,\nabla\cdot\mathbf{n}) = \gamma\,\nabla\kappa \;, \quad \kappa = \nabla\cdot\mathbf{n} = \mathrm{tr}\,\mathbf{K} \equiv K_{11} + K_{22}$$

The gradient of κ can cause pressure jump gradient, which may leads to flow.

• Surface normal & curvatures

Briefly review how to formulate $\mathbf{n}$ & $\nabla\cdot\mathbf{n}$ in cartesian coordinates. (x, y, z)

Cartesian coordinates (x, y, z)



$f(x,y,z) = C$
or
$z = h(x,y)$

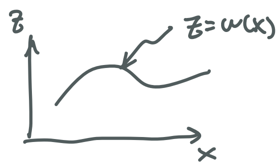
Define functional $F(x,y,z) = 0$ to describe the surface

The normal to the surface is

$$\underset{\sim}{n} = \frac{\nabla F}{|\nabla F|} = \frac{F_x \underset{\sim}{e}_x + F_y \underset{\sim}{e}_y + F_z \underset{\sim}{e}_z}{(F_x^2 + F_y^2 + F_z^2)^{1/2}}, \quad \text{where } F_x \text{ means } \frac{\partial F}{\partial}$$

The local curvature can be computed as $\kappa = \nabla \cdot \underset{\sim}{n}$

• Example 1. A 2D interface $z = h(x)$

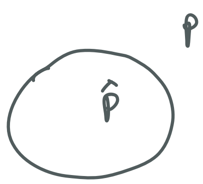


$$F = z - h(x) = 0$$

$$\underset{\sim}{n} = \frac{\nabla F}{|\nabla F|} = \frac{\underset{\sim}{e}_z - h_x \underset{\sim}{e}_x}{(1+h_x^2)^{1/2}}$$

$$\nabla \cdot \underset{\sim}{n} = \frac{-h_{xx}}{(1+h_x^2)^{1/2}} + \frac{h_x^2 h_{xx}}{(1+h_x^2)^{3/2}} = \frac{-h_{xx}}{(1+h_x^2)^{3/2}} \simeq -h_{xx} \quad \text{if } h_x^2 \ll 1 \text{ (Moderate rotation).}$$

• Example 2. A spherical interface $x^2 + y^2 + z^2 = R^2$



$$F = x^2 + y^2 + z^2 - R^2 = 0$$

$$\underset{\sim}{n} = \frac{\nabla F}{|\nabla F|} = \frac{x \underset{\sim}{e}_x + y \underset{\sim}{e}_y + z \underset{\sim}{e}_z}{(x^2 + y^2 + z^2)^{1/2}}$$

$$\nabla \cdot \underset{\sim}{n} = \frac{\partial n_x}{\partial x} + \frac{\partial n_y}{\partial y} + \frac{\partial n_z}{\partial z}$$

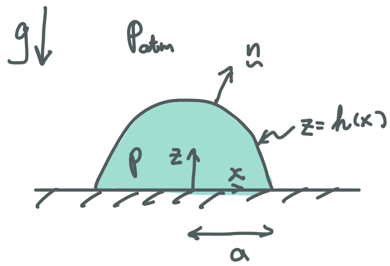
$$\frac{\partial n_x}{\partial x} = \frac{1}{(x^2+y^2+z^2)^{1/2}} + \frac{-x^2}{(x^2+y^2+z^2)^{3/2}} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$\rightarrow \nabla \cdot \underset{\sim}{n} = \frac{2R^2}{R^3} = \frac{2}{R}$$

$$\rightarrow \Delta p = \hat{p} - p = \gamma \nabla \cdot \underset{\sim}{n} = \frac{2\gamma}{R}$$

Fluid Statics

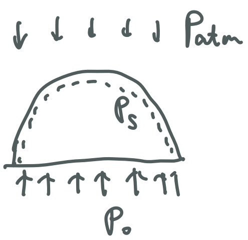
• Two-dimensional sessile drop



Across the L-V surface, using Laplace equation to have

$$P_s - P_{atm} = \gamma \nabla \cdot \underset{\sim}{n} = -\frac{\gamma h_{xx}}{(1+h_x^2)^{3/2}}$$

From the inner side of L-V surface to S-L interface, we have



$$P_0 = P_s + \rho g h \rightarrow \Delta P_0 = P_0 - P_{atm}$$

$$= -\frac{\gamma h_{xx}}{(1+h_x^2)^{3/2}} + \rho g h \equiv C.$$

We rewrite the ODE as

$$h_{xx} + \left(-\frac{\rho g h}{\gamma} + \frac{\Delta P_0}{\gamma}\right)(1+h_x^2)^{3/2} = 0$$

Non-dimensionalization

$$X = \frac{x}{a}, \quad H = \frac{h}{a}, \quad P = \frac{\Delta P a}{\gamma}, \quad B_0 = \frac{\rho g a^2}{\gamma}$$

Q: Is it reasonable to use "a" to rescale?

$$\Rightarrow \frac{a\,\partial^2 H}{a^2\,\partial X^2} + \Big(-\underbrace{\frac{\rho g a}{\gamma}}_{B_0/a} H + \underbrace{\frac{\Delta P}{\gamma}}_{P/a}\Big)\left[1+\left(\frac{\partial H}{\partial X}\right)^2\right]^{3/2} = 0$$

Governing equation

$$\frac{\partial^2 H}{\partial X^2} + (P - B_0 H)\left[1+\left(\frac{\partial H}{\partial X}\right)^2\right]^{3/2} = 0,$$

Bo – Bond number $\sim \dfrac{\text{Gravitational force}}{\text{Surface tension force}} = \dfrac{\rho g a^2}{\gamma} = \left(\dfrac{a}{l_c}\right)^2$

$$\begin{cases} Bo \gg 1, & a \gg l_c, \quad h_0 \sim l_c, \quad H_0 \sim \dfrac{l_c}{a} \\ Bo \ll 1, & a \ll l_c, \quad h_0 \sim a, \quad H_0 \sim 1 \end{cases}$$

Boundary conditions

$$\left.\frac{\partial H}{\partial X}\right|_{X=0} = 0, \quad H\Big|_{X=1} = 0, \quad \cos\theta = \frac{\gamma_{sv} - \gamma_{sl}}{\gamma}$$

Linearization by assuming $\theta \ll 1$, $h_x = H_x \ll 1$

$$\frac{\partial^2 H}{\partial X^2} - Bo H = -P,$$

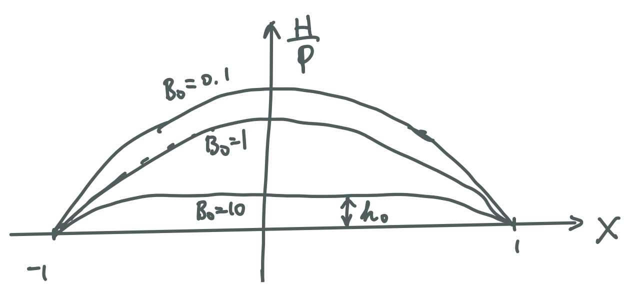
subject to

$$\left.\frac{\partial H}{\partial X}\right|_{X=0} = 0, \quad H\Big|_{X=1} = 0, \quad 1 - \frac{1}{2}\left(\frac{\partial H}{\partial X}\right)^2 = \cos\theta \quad (\theta \text{ close to } 0).$$

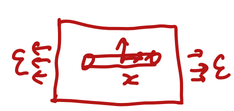
We then have

$$H_p = \frac{P}{Bo}, \quad H_h = Ae^{+Bo^{1/2}X} + Be^{-Bo^{1/2}X} = C\cosh(Bo^{1/2}X)$$

$$\rightarrow H = \frac{P}{Bo}\left[1 - \frac{\cosh(Bo^{1/2}X)}{\cosh(Bo^{1/2})}\right]$$, where P can be deterimed by contact angle condition.



This is also shear-lag solution to strain distribution in a fibre in a matrix



• $Bo \ll 1 \to$ ODE: $H_{xx} - \cancel{BoH} = -p$ $\left(H_{xx}/(BoH) \sim Bo^{-1} \gg 1 \right)$

$$\frac{H}{p} = \frac{1}{2}(1-x^2) \to h(x) = \frac{\Delta p}{2\gamma}(a^2 - x^2)$$

Check: $-\gamma h_{xx} = \Delta p$ ✓

What is Δp? Contact angle solution!

$$\left|\frac{\partial H}{\partial x}\right|_{x=1} = \left|h_x\right|_{x=a} = p = \frac{\Delta p a}{\gamma} = \sqrt{2}(1-\cos\theta)^{1/2}$$

$$\frac{h(x)}{a} = \frac{\sqrt{2}}{2}(1-\cos\theta)^{1/2}\left(1 - \frac{x^2}{a^2}\right)$$

Check: $h(0) = \frac{\sqrt{2}}{2}(1-\cos\theta)^{1/2} a \sim a$ ✓

• $Bo \gg 1$ $H_{xx} - BoH = -p$

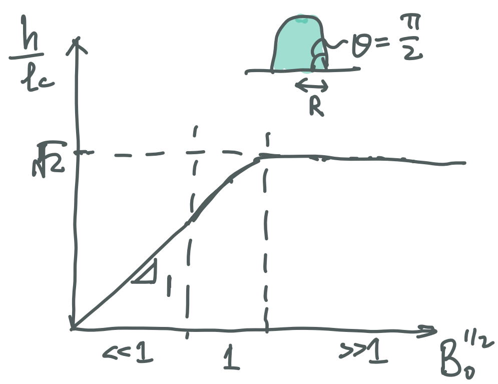
Away from the edge, $H_{xx}/BoH \sim Bo^{-1} \ll 1 \to H = \frac{p}{Bo}$

Near the edge, $H \to 0$, Need to consider H_x more formally

$$|H_x|_{x\to 1} = \frac{p\tanh(Bo^{1/2})}{Bo^{1/2}} = \frac{p}{Bo^{1/2}} \text{ as } Bo \to \infty$$

$$\to p = \sqrt{2}(1-\cos\theta)^{1/2} Bo^{1/2}, \quad H = \sqrt{2}(1-\cos\theta)^{1/2} Bo^{-1/2}$$

Check: $h(0) = \sqrt{2}(1-\cos\theta)^{1/2}\frac{a}{Bo^{1/2}} = \sqrt{2}(1-\cos\theta)^{1/2} l_c$ ✓



IN LAST CLASS, WE DREW THIS

$\theta = \frac{\pi}{2} \to$ A hemisphere

$\boxed{\text{Why } h(0) = \frac{\sqrt{2}}{2} a \text{ for } Bo \ll 1 \text{ this calculation, but the truth should be } h(0) = a\text{?}}$