Adhesion

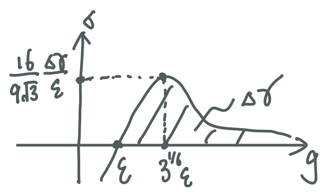
Adhesion can be thought of as a fracture mechanics problem. However, there are a number important length scale appearing in typical adhesion problems – making use of which can give some useful simplifications. We will mainly discuss the adhension between a sphere and a half plane here.

Bradley (1932) – Adhesion between a rigid sphere and a rigid half space

We first discuss interaction between two rigid half spaces following the L-J potential

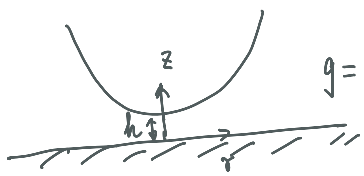
$$V = -\frac{A}{r^6} + \frac{B}{r^8}.$$

In lecture 2, we have derived the additive vdW forces between the two spaces:



$$\sigma(g) = \frac{8\Delta\gamma}{3\varepsilon}\left(\frac{\varepsilon^3}{g^3} - \frac{\varepsilon^9}{g^9}\right)$$

Now suppose a rigid sphere of radius $R \gg \varepsilon$ is placed near a rigid half-space such that the point of closet approach corresponds to $g = h$.



$$g = h + \frac{r^2}{2R}$$

The interaction force is now:

$$F = 2\pi\int_0^\infty r\,dr\,\sigma(g)$$

Since $dg = r\,dr/R$, we re-write

$$F = 2\pi R \underbrace{\int_h^{\infty} \sigma(g)\, dg}_{\text{Energy!}} = 2\pi R \times \frac{8\Delta\gamma}{3\varepsilon}\left(\frac{\varepsilon^3}{2h^2} - \frac{\varepsilon^9}{8h^8}\right) = 2\pi R\Delta\gamma\left[\frac{4}{3}\left(\frac{\varepsilon}{h}\right)^2 - \frac{1}{3}\left(\frac{\varepsilon}{h}\right)^8\right].$$

It is obvious that when $h = \varepsilon$, the force is maximized

$$F_{max} = 2\pi R\,\Delta\gamma$$

Think about why?

This conclusion applies for any contact problem with initial gap $g_0(r,\theta) = r^2 f(\theta)$.

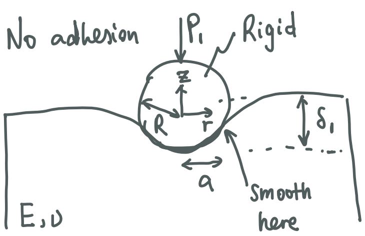
The JKR theory (1971)

JKR theory is concerned with the adhesive contact between elastic spheres.

Now there have been various ways to understand the JKR theory. The original paper Johnson et al (1971) and the paper by Maugis (1992) are recommended (reading the latter needs some knowledge of axisymmetric Fourier transform, i.e. Hankel transform, see Sneddon's book). However, here we give a rather rough introduction using some fracture mechanics concepts we learnt in last lecture.

The key idea is as follows:

No adhesion



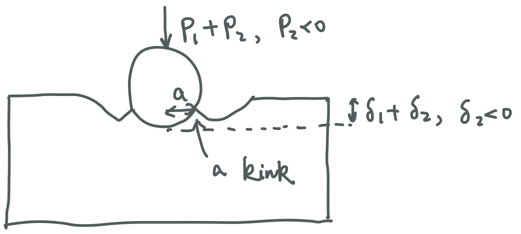
The solution to this bvp is

$$\frac{E}{1-\nu^2}$$

$$p_1(r) = \frac{2E^*\sqrt{a^2-r^2}}{\pi R}, \quad P_1 = \frac{4E^* a^3}{3R}, \quad \delta_1 = \frac{a^2}{R}$$

Think about why P-δ nonlinear?

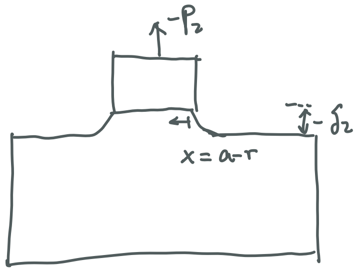
What if there is adhesion? Think of returning some P and δ.



What is this P_1 and its consequences?

Note that the contact radius is maintained!!!

This means you can pull the sphere back a little bit (causing a uniform upward displacement δ_2 in the contacted area). Note that the systems are superposable b/c a is fixed. This pulling-back is exactly a flat rigid cylindrical punch problem.



The solution to this bvp is

$$p_2(r) = \frac{P_2}{2\pi a\sqrt{a^2-r^2}}, \quad \delta_2 = \frac{P_2}{2E^* a}$$

The superposed P_2 is not arbitrary. The stress intensity cause by P_2 near the corner is given

$$p_2 = \frac{P_2}{2\pi a\sqrt{x(2a-x)}} = \frac{P_2}{2\pi a\sqrt{2ax}} \quad \text{as } x\to 0.$$

The related energy release rate can be computed (according to fracture mechanics)

$$K_I = \lim_{x\to 0} p\sqrt{2\pi x} = \frac{P_2}{2\sqrt{\pi a^3}}$$

$$G = \frac{K_I^2}{2E^*} = \frac{P_2^2}{8\pi E^* a^3} = \Delta\gamma$$

$$\rightarrow P_2 = -\sqrt{8\pi E^* a^3 \Delta\gamma}\,, \quad p_2(r) = -\sqrt{\frac{2E^* a\Delta\gamma}{\pi(a^2-r^2)}}\,, \quad \delta_2 = -\sqrt{\frac{2\pi a\Delta\gamma}{E^*}}$$

The solution to the adhesive contact between a rigid sphere and an elastic half plane is then

$$p(r) = p_1(r) + p_2(r) = \frac{2E^*\sqrt{a^2-r^2}}{\pi R} - \sqrt{\frac{2E^* a\Delta\gamma}{\pi(a^2-r^2)}}$$

$$P = P_1 + P_2 = \frac{4E^* a^3}{3R} - \sqrt{8\pi E^* a^3 \Delta\gamma}$$

$$\delta = \delta_1 + \delta_2 = \frac{a^2}{R} - \sqrt{\frac{2\pi a\Delta\gamma}{E^*}}$$

Non-dimensionalization. Natural to take

$$\bar{P} = \frac{P}{\pi R\Delta\gamma}\,, \quad \Delta = \frac{\delta}{R}\,, \quad \Lambda = \frac{a}{R}$$

so that

$$\bar{P} = \frac{4E^* R}{3\pi\Delta\gamma}\Lambda^3 - \left(\frac{8E^* R}{\pi\Delta\gamma}\right)^{1/2}\Lambda^{3/2}$$

$$\Delta = \Lambda^2 - \left(\frac{2\pi\Delta\gamma}{E^* R}\right)^{1/2}\Lambda^{1/2} \quad \text{— Not very successful.}$$

Some thoughts about the contact area and indentation depth. In the regime of interest:

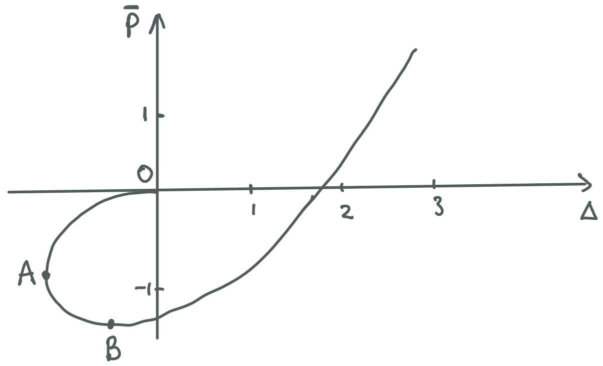
$$\delta \sim \frac{a^2}{R} \sim \left(\frac{a\Delta\gamma}{E^*}\right)^{1/2} \rightarrow \frac{a}{R} \sim \left(\frac{\Delta\gamma}{E^* R}\right)^{1/3}, \quad \frac{\delta}{R} \sim \left(\frac{\Delta\gamma}{E^* R}\right)^{2/3}$$

Then define

$$\beta = \left(\frac{E^* R}{\Delta\gamma}\right)^{1/3}, \quad \Lambda = \frac{a}{R}\cdot\beta\,, \quad \Delta = \frac{\delta}{R}\beta^2$$

so that

$$\bar{P} = \frac{4}{3\pi}\Lambda^3 - \frac{4}{\sqrt{2\pi}}\Lambda^{3/2}\,, \quad \Delta = \Lambda^2 - \sqrt{2\pi\Lambda}$$



• At B, $\frac{d\bar{P}}{d\Lambda} = \frac{4}{\pi}\Lambda^2 - \frac{6}{\sqrt{2}}\Lambda^{1/2} = 0 \rightarrow \Lambda_B = \left(\frac{9\pi}{8}\right)^{1/3}$

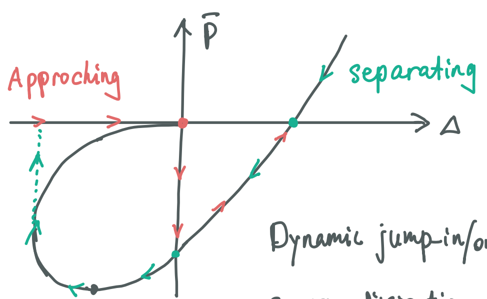
$\bar{P}_B = -\frac{3}{2}$

→ We need apply a tensile force $F = -\bar{P}_B \times \pi R \Delta\gamma = \frac{3}{2}\pi R \Delta\gamma$ to separate the sphere from the half-space.

→ The radius of contact area at this instant is $a = \Lambda_B \times \beta R = \left(\frac{9\pi R^2 \Delta\gamma}{8E^*}\right)^{1/3}$

• At A, $\frac{d\Delta}{d\Lambda} = 2\Lambda - \frac{1}{2}\sqrt{\frac{2\pi}{\Lambda}} = 0 \rightarrow \Lambda_A = \left(\frac{\pi}{8}\right)^{1/3}$, $\Delta_A = -\frac{3}{4}\pi^{2/3}$

A number of interesting aspects arise here. First, the stability. It can be proved (similar approach we used in last lecture) that for displacement-control, the stability is retained unit the point A while point B for load control.



Dynamic jump-in/out processes lead to energy dissipation in the form of elastodynamic waves.

The Tabor parameter

A particular conclusion of JKR theory is that the maximum force to separate the rigid sphere and the elastic half space is

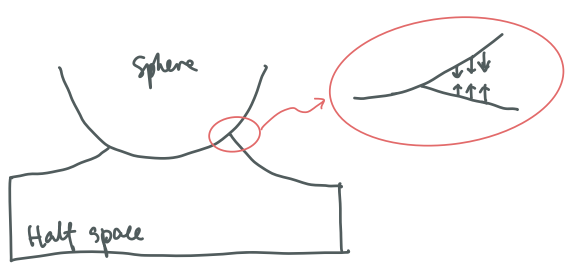
$$F_{max}^{JKR} = \frac{3}{2}\Delta\gamma R \quad (\text{independent of } E^*).$$

While Bradley analysis for rigid sphere and half space is

$$F_{max}^{Bradley} = 2\Delta\gamma R.$$

Note that problem 1 in HW2 gives $F^{Capillary} = 4\gamma R = 2\Delta\gamma R$ as well (interestingly).

Why $F_{max}^{JKR}\Big|_{E^*\to\infty} \neq F_{max}^{Bradley}$? The interaction between bodies outside the contact area is not considered in JKR.

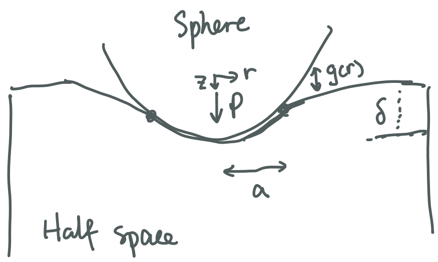


Several theories to bridge JKR and Bradley:

- Derjaguin et al. (1975) DMT theory
- Maugis (1992) Maugis - Dugdale theory
- Greenwood (1997) "Self-Consistent model"

- DMT theory (A farewell?)

It essentially combines Hertzian contact solution (no adhesion) and σ-g law.



Assuming Hertizan contact so that

$$u_z(r) = \frac{(2a^2-r^2)}{\pi R}\,\text{Arcsin}\left(\frac{a}{r}\right) + \frac{a\sqrt{r^2-a^2}}{\pi R}, \quad r>a$$

upward displacement

The adhesive fore is computed in the region outside the contact:

$$F_1 = 2\pi\int_a^{\infty} \sigma(g)\, r\,dr, \quad g = \underset{\text{equilibrium.}}{\xi} + \frac{r^2}{2R} - \delta + u_z(r)$$

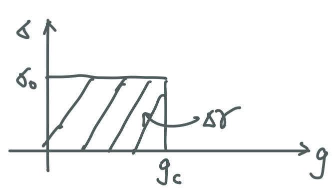
So the total indenting force is $P = \underset{\text{Hertz}}{P_H} - F_1$

Minis for attractive adhesion.

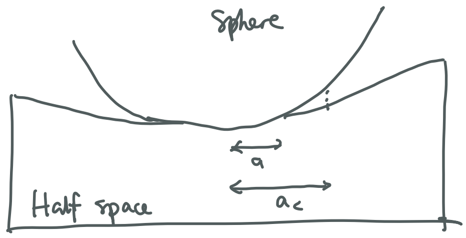
The DMT theory is rough, particularly in the consideration of elastic deformation in both $[0,a]$ and $[a,\infty)$. However, somehow it predeets the same pull off force as Bradley.

- Maugis' solution

Maugis (1992) made the further simplification of using a Dugdale cohesive zone approximation (Dugdale, 1960) and solved the bvp with mixed boundaries.



Dugdale cohesive law.

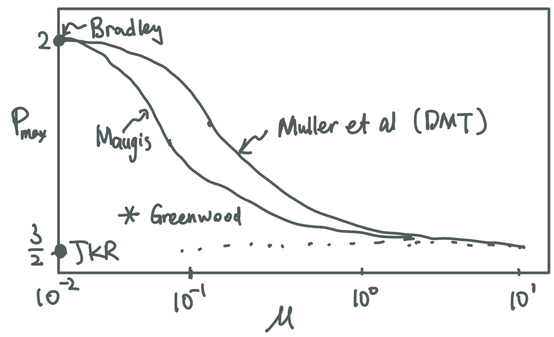


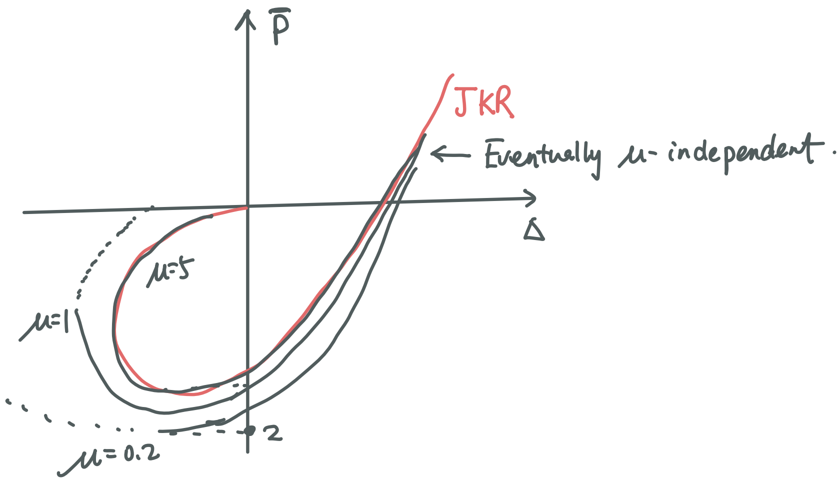
- $0 < r < a$, prescribed displacement (by the indenter)
- $a < r < a_c$, $\sigma_{zz} = +\sigma_0$ (at $r = a_c$, $g = g_c$)
- $r > a_c$, $\sigma_{zz} = 0$

An important parameter arises here defined by Tabor

$$\mu = \left(\frac{R \, \Delta\gamma^2}{E^{*2} \varepsilon^3} \right)^{1/3}$$

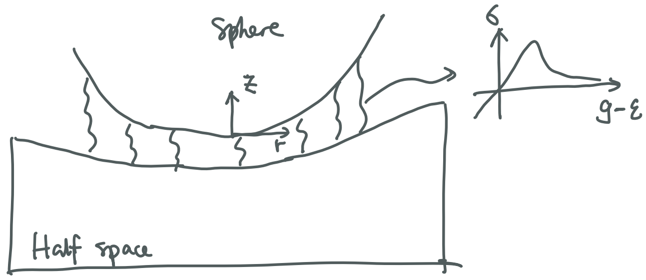
(to be discussed soon).

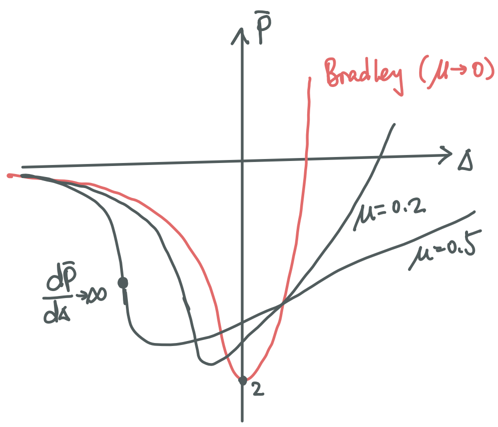


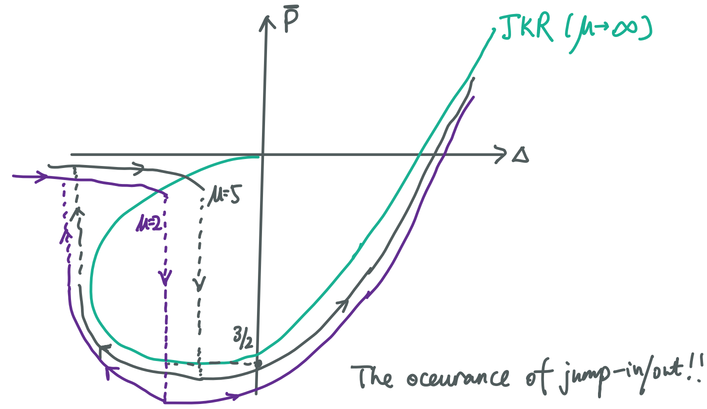


- Greenwood's solution

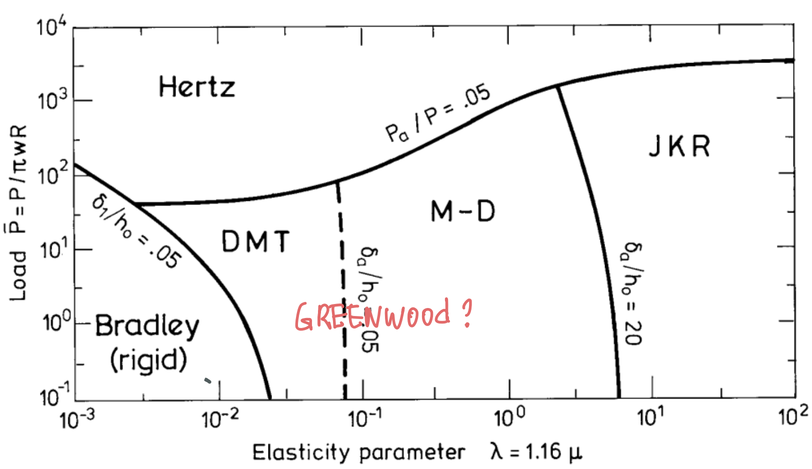
Greenwood (1997) directly solve the bvp with the consideration of intermolecular forces.







Let us make a summary here via an adhesion map drawn by Greenwood and Johnson (1997).



Finally how to understand the Tabor parameter (1977) that can be used to bridge different models or to know whether small-scale σ-g laws to be considered?

- Point of view of lengths. Recall we used $\Delta = \frac{\delta}{R}\beta^2$ to define rescaled vertical indentation displacement. Natural to have a parameter by taking $\delta = \varepsilon$. Indeed

$$\mu = \frac{R}{\beta^2 \varepsilon} = \left(\frac{R\Delta\gamma}{E^{*2}\varepsilon^3}\right)^{1/3}, \quad \beta = \left(\frac{E^* R}{\Delta\gamma}\right)^{1/3} \quad \text{so that} \quad \begin{cases} \mu \gg 1 \rightarrow \Delta_{\delta=\varepsilon} \ll 1 \\ \mu \ll 1 \rightarrow \Delta_{\delta=\varepsilon} \gg 1 \\ \text{Regime of interest } \Delta \sim 1 \end{cases}$$

- Point of view of forces. Again from JKR theory, we have

$$a = \left(\frac{9\pi R^2 \Delta\gamma}{8E^*}\right)^{1/3}.$$

We want to use this to compare to a horizontal lengthscale to obtain μ.

Note that the stress field near the contact line is given as

$$\sigma \sim \frac{K_I}{\sqrt{2\pi r}} \sim \sqrt{\frac{E^* \Delta\gamma}{\pi r}}.$$

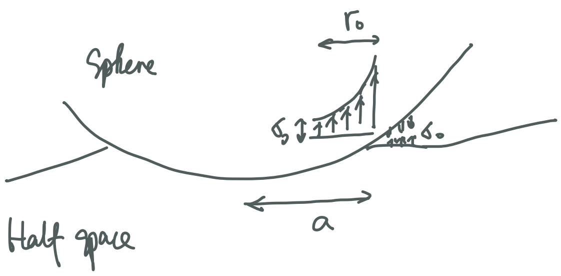
JKR did not consider the adhesive interactions outside the contact area in which $\sigma \sim \frac{\Delta\gamma}{\varepsilon}$. An natural horizontal length (in the contact area) appears:

$$\sqrt{\frac{E^* \Delta\gamma}{\pi r_0}} \sim \frac{\Delta\gamma}{\varepsilon} \rightarrow r_0 \sim \frac{E^*}{\Delta\gamma}\varepsilon^2$$

It can be shown that

$$\frac{a}{r_0} \sim \left(\frac{R\Delta\gamma^2}{E^{*2}\varepsilon^3}\right)^{2/3} = \mu^2$$

Physical picture:



$\mu \gg 1 \rightarrow a \gg r_0$

$\mu \ll 1 \rightarrow a \ll r_0$