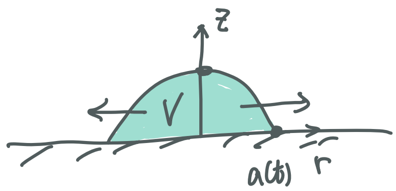
• Evolution of a "large" drop

• The problem



$a \sim t^{1/8}$ in last lecture

By "large", I mean $Bo \gg 1$ so that the shape of the drop evolves as

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu} \frac{1}{r}\frac{\partial}{\partial r}\left(r h^3 \frac{\partial h}{\partial r}\right) \qquad \frac{\partial h}{\partial t} = \frac{1}{3\mu}\nabla\cdot\left(h^3 \nabla p\right)$$

Subject to

$$h(x,0) = h_0(x)$$

$$\left.\begin{aligned} & h(a,t) = 0 \\ & \left.\frac{\partial h}{\partial x}\right|_0 = 0 \\ & 2\pi\int_0^a r h \, dr = V \end{aligned}\right\} \text{ 2nd order derivative in space + unknown } a(t)$$

• Non-dimensionalization

$$H = h/\ell, \quad R = r/\ell, \quad \ell = V^{1/3}, \quad T = t/t^*$$

Using these rescalings, we have

$$\frac{\partial H}{\partial T} = \frac{\rho g t^* \ell}{3\mu} \frac{1}{R}\frac{\partial}{\partial R}\left(R H^3 \frac{\partial H}{\partial R}\right)$$

Naturally to choose $t^* = \frac{3\mu}{\rho g \ell} \sim \frac{[N/m^2 \cdot s]}{N/m^3 \cdot m} \sim [s]$ ✓ so that

$$\frac{\partial H}{\partial T} = \frac{1}{R}\frac{\partial}{\partial R}\left(R H^3 \frac{\partial H}{\partial R}\right)$$

• Similarity solution

Look for similarity solution of the form

$$H(T,\xi) = T^{\alpha} f(\xi) \quad \text{where } \xi = \frac{R}{T^{\beta}}.$$

The derivative of $H(T,\xi)$ w.r.t. T

$$-\beta\frac{R}{T^{\beta+1}} = -\beta\xi/T$$

$$\frac{\partial H}{\partial T} = \alpha T^{\alpha-1} f(\xi) + T^{\alpha}\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial T}$$

$$= \alpha T^{\alpha-1} f(\xi) - \beta\xi T^{\alpha-1}\frac{\partial f}{\partial \xi}$$

The derivative w.r.t. R

$$\frac{\partial}{\partial R} = \frac{\partial}{\partial \xi}\cdot\underbrace{\frac{\partial \xi}{\partial R}}_{T^{-\beta}} = T^{-\beta}\frac{\partial}{\partial \xi}$$

$$\frac{\partial H}{\partial R} = T^{\alpha-\beta}\frac{\partial f}{\partial \xi}$$

The PDE now can be rewritten as

$$T^{\alpha-1}(\alpha f - \beta\xi f') = T^{-\beta}\frac{1}{\xi}T^{-\beta}\frac{\partial}{\partial \xi}\left(T^{\beta}\xi T^{3\alpha} f^3 T^{\alpha-\beta} f'\right)$$

$$= T^{4\alpha-2\beta}\frac{1}{\xi}(\xi f^3 f')'$$

Hope $f(\xi)$ independent of T in the similarity solution postulate, we must have

$$3\alpha - 2\beta + 1 = 0$$

Consider next the conservation of total mass of the drop.

$$2\pi\int_0^{A(t)} HR\,dR = 1 \quad , \text{where } A(t) = a(t)/V^{1/3}$$

Assume that $A(t) = \xi_0 T^{\beta}$ so that

$$2\pi \int_0^{\xi_0} f \xi d\xi \times T^{\alpha+2\beta} = 1 ,$$

which requires

$$\alpha = -2\beta \rightarrow \beta = \frac{1}{8}, \ \alpha = -\frac{1}{4}, \quad A(t) = \xi_0 T^{\frac{1}{8}}$$ what is ξ_0?

Back to the similarity equation

$$-\frac{1}{4}\xi f - \frac{1}{8}\xi^2 f' = (\xi f^3 f')' .$$

which can be regrouped to be

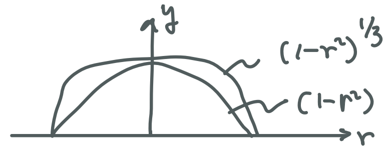
$$\frac{1}{8}(\xi^2 f)' + (\xi f^3 f')' = 0$$

$$\rightarrow \frac{1}{8}\xi^2 f + \xi f^3 f' = \cancel{\text{Cnst.}} \rightarrow 0$$ $f' \to 0$, f finite as $\xi \to 0$

$$\rightarrow \frac{1}{8}\xi + f^2 f' = 0 \quad \text{with } f(\xi_0) = 0$$

Mathematica gives

$$\boxed{f = \left(\frac{3}{16}\right)^{1/3} (\xi_0^2 - \xi^2)^{1/3}}$$



(what happened near the contact line?)

The specific ξ_0 is selected to satisfy

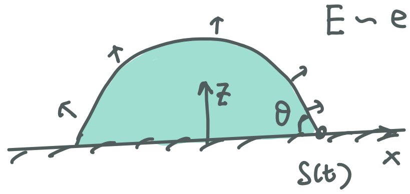
$$2\pi \int_0^{\xi_0} \left(\frac{3}{16}\right)^{1/3} (\xi_0^2 - \xi^2)^{1/3} \xi d\xi = 1 \rightarrow \xi_0 = \left(\frac{2^{10}}{3^4 \pi^3}\right)^{1/8}$$

$$\Rightarrow a(t) = \xi_0 \left(\frac{t}{t_*}\right)^{1/8} l = \underbrace{\left(\frac{2^{10}}{3^5 \pi^3}\right)^{1/8}}_{0.89} \left(\frac{\rho g V^3}{\mu}\right)^{1/8} t^{1/8}$$

$$h(t) = 0.70 \left(\frac{\mu V}{\rho g}\right)^{1/4} t^{-1/4} \left[1 - \left(\frac{r}{a}\right)^2\right]^{1/3}$$

• Evaporation of a small drop ($Ca \ll 1$, $Bo \ll 1$)

The problem



$E \sim$ evaporation rate [m/s]

Suppose that the evaporation is uniform across the surface of the blob (in pratice there will be more evaporation from the edges than from the center).

Now the total mass is not constant:

$$\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = -E$$

(What about drainage?)

Using no-slip at the drop-solid interface and no-shear condt. at the drop-air interface as well as arbitrary ℓ, U_0, $t^* = \ell/U$, we have derived

$$H_T + \left[\frac{1}{3Ca} H^3 \left(H_{xxx} - \cancel{B_0 H_x}^{\,0}\right)\right]_x = -\cancel{\frac{E}{U_0}}^{\,1} .$$

Naturally choose $U_0 = E$, $Ca = \frac{\mu E}{\gamma}$. We assume $K \gg E$ so that $\theta \equiv \theta_0$

Dynamic contact line speed.

The dynamics is driven solely by evaporation!!!

$$\boxed{3Ca\, H_T + \left(H^3 H_{xxx}\right)_x = -3Ca}$$

Let us seek solution of form

$$H = H_0 + Ca\, H_1 + Ca^2 H_2 \cdots\cdots$$

$$H_{xxx} = H_{0,xxx} + Ca\, H_{1,xxx} + Ca^2 H_{2,xxx}$$

$$H^3 = H_0^3 + 3Ca\, H_0^2 H_1 + 3\, Ca^2 H_0 H_1^2 + \cdots$$

$$H^3 H_{xxx} = H_0^3 H_{0,xxx} + 3Ca\, H_0^3 H_{1,xxx} + 3\,Ca\, H_0^2 H_1 H_{0,xxx} + O(Ca^2).$$

At leading order $O(Ca^0)$

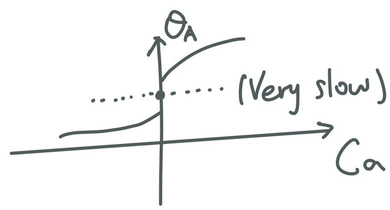
$$\left(H_0^3 H_{0,xxx} \right)_x = 0$$ Quasi-static!

Subject to

$H_{0,x}(0) = 0$ → Zero slope
$H_{0,xxx}(0) = 0$ → Zero flux
} Symmetry condition.

$H_0(S) = 0$

$H_{0,x}(S) = -\theta_0$



The solution (identical to that we derived in last lecture!)

$$H = \frac{\theta_0}{2S}(S^2 - x^2)$$

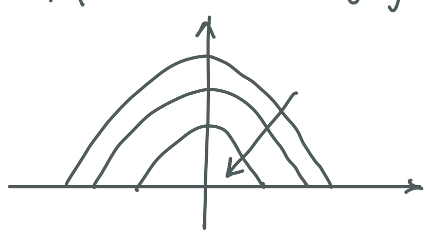
To compute S, we need to return the evaporation equation

0 - Physically no flux at either end.

$$\int_{-S}^{S} dx \left[3Ca\, H_T + \cancel{\left(H^3 H_{xxx} \right)_x} \right] = -3Ca \times 2S$$

$$\frac{d}{dT}\int_{-S}^{S} H\, dx = -2S = \frac{d}{dT}\left(\frac{\theta_0}{2S}\left(S^2 x - \frac{1}{3}x^3 \right)\Big|_{-S}^{S} \right) = \frac{d}{dT}\left(\frac{2}{3}\theta_0 S^2 \right) = \frac{4\theta_0 S \dot{S}}{3}$$

$$\Rightarrow \dot{S} = -\frac{3}{2\theta_0}$$



But $\bar{U} = \frac{Q_0}{H_0} = H_0^3 H_{0,xxx} = 0$?
(Average velocity) ☆

At first order $O(Ca)$

$$3Ca\left(H_{0,T} + \cancel{Ca H_{1,T}}^{\,0}\right) + \left(\cancel{H_0^3 H_{0,xxx}}^{\text{Auto-satisfied.}} + 3Ca\, H_0^3 H_{1,xxx} + 3Ca\, H_0^2 H_1 \cancel{H_{0,xxx}}^{\,0}\right)_x = -3Ca$$

$$\boxed{H_{0,T} + \left(H_0^3 H_{1,xxx}\right)_x = -1}$$

The flow field is given by $\bar{u} = \cancel{\frac{q_0}{Ca}}^{\,0} + q_1 + Ca\, q_2 \sim q_1 = H_0^2 H_{1,xxx}$

$$\rightarrow \quad (H_0 q_1)_x = -1 - H_{0,T}$$

$$= -1 - \frac{1}{2}\theta_0 \dot{S} - \frac{1}{2}\theta_0 x^2 \frac{1}{S^2}\dot{S}$$

$$= -\frac{1}{4} + \frac{3}{4}\frac{x^2}{S^2}$$

We have

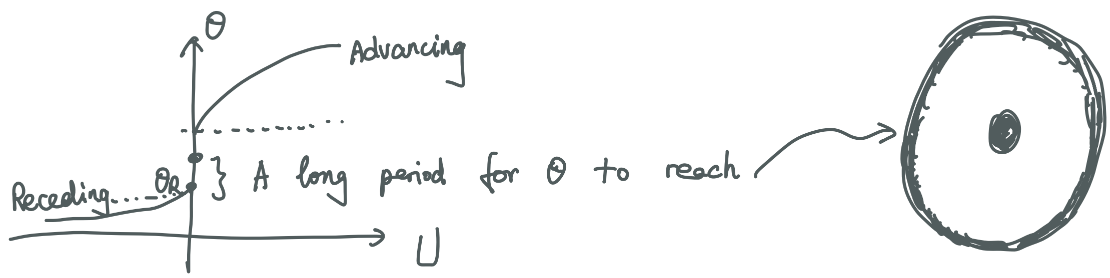
$$\bar{u} \sim q_1 = \frac{1}{H_0}\int_0^x \left(-\frac{1}{4} + \frac{3}{4}\frac{x^2}{S^2}\right)dx + \text{Const}$$

$$= \frac{2S}{\theta_0}\frac{1}{S^2 - x^2}\left(-\frac{1}{4}x + \frac{1}{4}\frac{x^3}{S^2}\right)$$

$$= -\frac{1}{2\theta_0}\frac{x}{S}$$

where we have used $\bar{u}(0) = 0$. The flow is inward from the contact line to the center (Coffee eye!!!)

- Pinned contact line



The steady solution is

To be determined.

$$H = \frac{3A(t)}{4S^3}\left(S^2 - x^2\right), \quad S \text{ is "fixed"}$$

$$\theta = \frac{3A(t)}{2S^2} \quad (\downarrow)$$

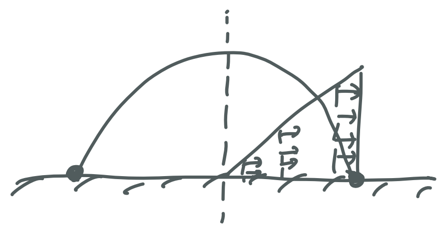
$$\dot{A} = \frac{d}{dt}\int_{-S}^{S} H\,dx = -2S$$

$$q_1 = \frac{1}{H_0}\int_0^x \left(-1 - H_{0,T}\right) dx$$

$$= \frac{4S^3}{3A(S^2-x^2)}\int_0^x \left[-1 + \frac{6S}{4S^3}(S^2-x^2)\right] dx$$

The velocity field is now given by

$$\bar{u} \sim q_1 = \frac{2Sx}{3A} \quad \text{(Outward)}$$



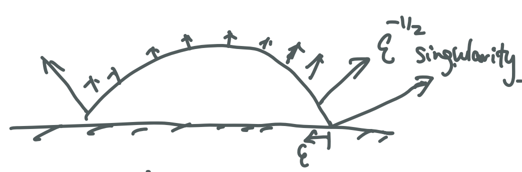
See R.D. Deegan et al Nature (1998) for a detailed model of this problem

(incorporating non-uniform evaporation rates).

$$E \sim (x-S)^{-\lambda}, \quad \lambda = (\pi - 2\theta)/(2\pi - 2\theta) \to 1/2.$$

a result of $\frac{d\phi}{dt} \propto \nabla^2\phi \simeq 0$ (Quasi-static Again).

- Gradient of surface tension



Evaporation leads to surface cooling, which is mos pronounced near the edge. ∇T or $\nabla C \to \nabla\gamma \to$ Shear (Marangoni) stress $\to$ radial flow

• Marangoni flow

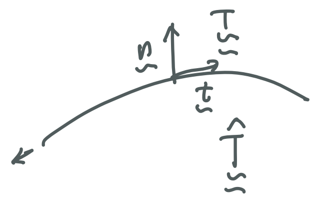
Surface-tension flow is driven by

$$\nabla p = \nabla(\gamma \kappa) = \gamma \nabla \kappa + \nabla \gamma \, \kappa$$

where $\nabla \gamma \ll \gamma$ typically. Before we used $\nabla\gamma = 0$, now let's see what occurs when $\nabla \gamma \neq 0$.

In fluid statics, we showed the interfacial stress balance equation

$$\underset{\sim}{n} \cdot \left(\underset{\approx}{T} - \underset{\approx}{\hat{T}} \right) + \nabla \gamma - \gamma \underset{\sim}{n} \left(\nabla \cdot \underset{\sim}{n} \right) = 0$$



• Normal direction $\rightarrow$ $\underbrace{\Delta p = \hat{p} - p = \gamma \nabla \cdot \underset{\sim}{n}}$

Remain appropriate when $\nabla\gamma \neq 0$ for thin films. Why?

• Tangential direction

$$\underset{\sim}{n} \cdot \left(\underset{\approx}{T} - \underset{\approx}{\hat{T}} \right) \cdot \underset{\sim}{t} + \nabla \gamma \, \underset{\sim}{t} - \gamma \left(\cancelto{0}{\underset{\sim}{n} \cdot \underset{\sim}{t}} \right) \left(\nabla \cdot \underset{\sim}{n} \right) = 0$$

Stress tensors write

$$\underset{\approx}{T} = -p \underset{\approx}{I} + \underbrace{2 \mu_{Air} \underset{\approx}{\epsilon}_{Air}} , \quad \underset{\approx}{\epsilon} = \frac{1}{2} \left(\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T \right) - \frac{1}{3} Tr(\underset{\approx}{\epsilon}) \underset{\approx}{I}$$

$\underset{\approx}{C}$, deviatoric stress

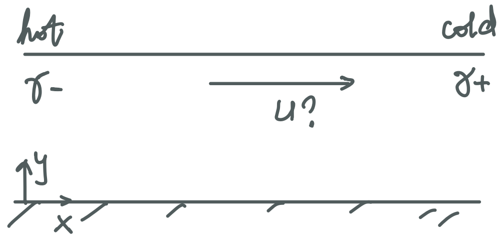
rate of strain tensor

$$\underset{\approx}{\hat{T}} = -\hat{p} \underset{\approx}{I} + 2 \mu \underset{\approx}{\epsilon}$$

$$\Rightarrow \quad \underset{\sim}{n} \cdot \left[(\hat{p} - p) \underset{\approx}{I} + 2 \left(\cancelto{\text{Negligible}}{\mu_{Air} \underset{\approx}{\epsilon}_{Air}} - \mu \underset{\approx}{\epsilon} \right) \right] \cdot \underset{\sim}{t} = -\nabla \gamma \, \underset{\sim}{t}$$

$$\Rightarrow \quad \underset{\sim}{n} \cdot \underset{\approx}{C} \cdot \underset{\sim}{t} = \nabla \gamma \, \underset{\sim}{t}$$

2D simplification



$$\gamma = \gamma_0 \left(1 - \frac{T}{T_c}\right)^n, \quad n \sim 11/9.$$

$$T = T_0 - Gx, \quad \frac{\partial T}{\partial x} = G$$

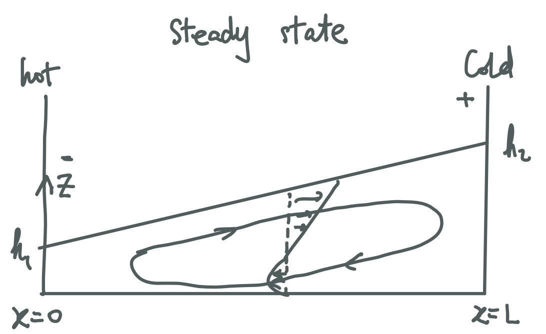
$$\Rightarrow \frac{\partial \gamma}{\partial x} = \frac{\partial \gamma}{\partial T} \cdot \frac{\partial T}{\partial x} \simeq + \frac{\gamma_0 G}{T_c} \quad (n \simeq 1)$$

$$\underset{\sim}{n} \cdot \underset{\approx}{\tau} \cdot \underset{\sim}{t} = \underbrace{[0 \quad \underset{\sim}{e_y} \quad 0]}_{\text{small slopes}} \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \begin{bmatrix} \underset{\sim}{e_x} \\ 0 \\ 0 \end{bmatrix} = \tau_{xy}$$

$$\nabla \gamma \cdot \underset{\sim}{t} = \left(\frac{\partial \gamma}{\partial x} \underset{\sim}{e_x} + \frac{\partial \gamma}{\partial y} \underset{\sim}{e_y}\right) \cdot \left(\underset{\sim}{e_x} \quad 0\right) = \frac{\partial \gamma}{\partial x}$$

$$\Rightarrow \mu \left(\frac{\partial u}{\partial y} + \underbrace{\cancel{\frac{\partial v}{\partial x}}}_{\text{High-order}}\right) = + \frac{\gamma_0 G}{T_c}, \quad \text{i.e.,} \quad \frac{\partial u}{\partial y} = + \frac{\gamma_0 G}{\mu T_c}$$

Example : Shallow pan problem



$$T = T_0 - Gx$$

$$\rightarrow \frac{\partial u}{\partial z} = \frac{\gamma_0 G}{\mu T_c} \quad (\text{On the surface})$$

The steady state flow is unidirectional ($h \ll L$)

$$\frac{\partial p}{\partial z} = -\rho g \rightarrow p = \rho g (h - z) - \overbrace{\gamma h_{xx}}^{f(x)}$$

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2} \rightarrow u = \frac{1}{2\mu}\frac{\partial p}{\partial x} z^2 + C_1 z + C_2$$

Boundary conditions are

$$u = 0 \quad \text{at} \quad z = 0 \rightarrow C_2 = 0$$

$$\frac{\partial u}{\partial y} = \frac{\gamma_0 G}{\mu T_c} \quad \text{at } z = h \rightarrow u = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(z^2 - 2zh\right) + \underbrace{\frac{\gamma_0 G}{\mu T_c} z}_{\text{Correction}}$$

Since the system is steady-state, we require

$$Q = \int_0^h u\, dz = 0 \rightarrow -\frac{1}{3\mu}\frac{\partial p}{\partial x} h^3 + \frac{1}{2}\frac{\gamma_0 G}{\mu T_c} h^2 = 0 \rightarrow \frac{\partial p}{\partial x} = \frac{3}{2}\frac{\gamma_0 G}{T_c}\frac{1}{h}$$

The velocity field reads

$$\boxed{u = \frac{\gamma G h}{2\mu T_c}\left[\frac{3}{2}\left(\frac{z}{h}\right)^2 - \frac{z}{h}\right]}$$

Finally. Let us examine the shape of the film. For example, when gravitational force dominate over capillary force.

$$\frac{\partial p}{\partial x} = \frac{3}{2}\frac{\gamma_0 G}{T_c}\frac{1}{h} = \rho g \frac{dh}{dx} \rightarrow \boxed{h^2(x) - h_1^2 = \frac{3\gamma_0 G}{\rho g T_c} x} \rightarrow$$

Note that this is appropriate only when

$$h \sim \left(\frac{3\gamma_0 G}{\rho g T_c} x + h_1^2\right)^{1/2}$$

$$\rho g h \gg \gamma h_{xx} \rightarrow \rho g h_1 \gg \underbrace{\gamma_0 \left(\frac{\gamma_0 G}{\rho g T_c}\right)^2 \Big/ h_1^3}$$ ← Maximum κ given at $x=0$.

$$\text{i.e.} \quad h_1 \gg l_c^{3/2} \times \underbrace{\left(\frac{G}{T_c}\right)^{1/2}}_{\text{Interesting.}} \sim \left(\frac{\Delta T}{T_c}\right)^{1/2} \times l_c \times \left(\frac{l_c}{L}\right)^{1/2}$$