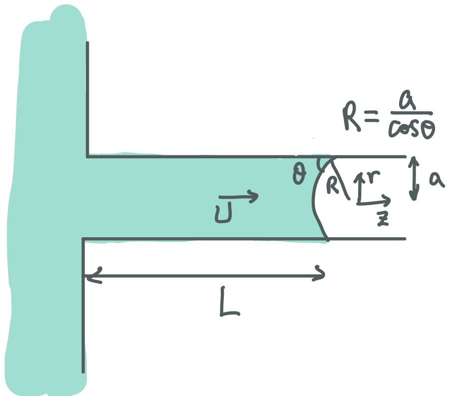
Moving contact lines

• Wicking problem (ex. 1999 Ig Nobel prize)

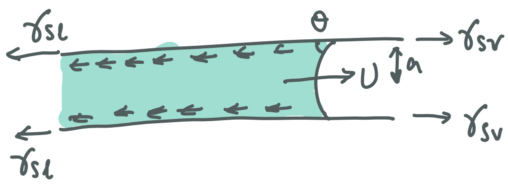
Let us review some basic concepts by working on this interesting question:



How fast does the front grow?

Let's say $Bo=\frac{\rho g a^2}{\gamma}$, $Ca=\frac{\mu U}{\gamma} \ll 0$ so it is a matter viscosity and Capillarity.

• A scaling argument of force balance



$\gamma_{sv}-\gamma_{sl}=\gamma\cos\theta$

Viscous force $\sim \mu\frac{U}{a}\times aL=\mu LU$

Capillar force $=\begin{cases} 2\pi a\times\gamma\cos\theta \\ \text{or} \\ \pi a^2\times 2\gamma/R \end{cases} \sim a\gamma\cos\theta$

$\rightarrow LU=\frac{1}{2}\frac{d}{dt}(L^2)\sim\frac{a\gamma\cos\theta}{\mu}$

i.e. $L\sim\left(\frac{a\gamma\cos\theta}{\mu}\right)^{1/2}t^{1/2}$

• A formal analysis assuming fully developed Poiseuille flow $\left(\frac{\partial u}{\partial z}=0\right)$

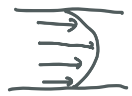
$$\underbrace{\frac{\partial p}{\partial z}}_{\text{function of } z}=\mu\nabla^2 u=\underbrace{\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)}_{\text{function of } r}+\cancel{\mu\frac{\partial^2 u}{\partial z^2}}^{\,0}$$

$$\rightarrow \frac{\partial p}{\partial z} \equiv -\frac{2\gamma\cos\theta}{aL}$$

$$\rightarrow u = -\frac{\gamma\cos\theta}{2\mu a L} r^2 + A\ln r + B$$

Use $\frac{\partial u}{\partial r}\Big|_0 = 0$, $u\big|_a = 0$ to show

$$u = \frac{\gamma\cos\theta\, a}{2\mu L}\left(1 - \frac{r^2}{a^2}\right)$$



Finally the flow rate can be calculated

$$Q = 2\pi\int_0^a u r\,dr = \frac{\pi\gamma\cos\theta\, a^3}{4\mu L} = \pi a^2 U$$

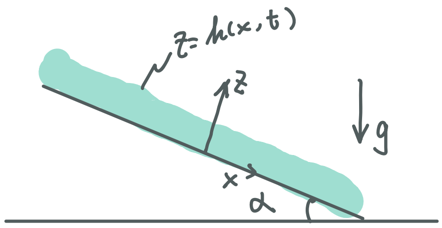
$$\Rightarrow LU = \frac{1}{2}\frac{d}{dt}L^2 = \frac{a\gamma\cos\theta}{4\mu}, \text{ i.e. } L = \left(\frac{a\gamma\cos\theta}{2\mu}\right)^{1/2} t^{1/2}$$

Washburn's equation. (1921)

Front slows down due to increasing viscous dissipation with increasing column length!

• Free surface flow down an inclined plane.

Let's first consider a problem to i) include gravity properly and 2) introduce a slip boundary.



$$\underset{\sim}{g} = g(\sin\alpha\, \underset{\sim}{e}_x - \cos\alpha\, \underset{\sim}{e}_z)$$

Recall:

$$\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0 \,, \quad Q = \int_0^h u\, dz$$

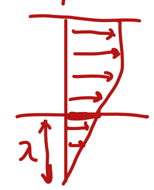
$$\frac{\partial p}{\partial z} = -\rho g \cos\alpha \rightarrow p = -\gamma h_{xx} + \rho g (h - z)\cos\alpha$$

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2} + \rho g \sin\alpha \rightarrow \mu \frac{\partial^2 u}{\partial z^2} = \underbrace{-\gamma h_{xxx} + \rho g \cos\alpha\, h_x - \rho g \sin\alpha}_{f}$$

$$\rightarrow u = \frac{f}{2\mu} z^2 + C_1 z + C_2$$

Use boundary conditions

Interpretation of λ



slip length.

$$\frac{\partial u}{\partial z} = 0 \quad \text{at} \quad z = h \quad (\text{No-shear}) \rightarrow C_1 = -\frac{f}{\mu} h$$

$$u = \lambda \frac{\partial u}{\partial z} \quad \text{at} \quad z = 0 \quad (\text{slip}) \rightarrow C_2 = \lambda C_1$$

$$\rightarrow u = \frac{f}{2\mu}\left(z^2 - 2hz - 2\lambda h \right)$$

Now we have the governing equation

$$Q = \int_0^h u\, dz = -\frac{f}{3\mu} h^3 - \frac{f}{\mu}\lambda h^2$$

$$\frac{\partial h}{\partial t} = -\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{\mu}\left(\frac{1}{3} h^3 + \lambda h^2 \right)\left(-\gamma h_{xxx} + \rho g \cos\alpha\, h_x - \rho g \sin\alpha \right) \right]$$

No-dimensionalization

$$H = h/\ell \,, \quad X = x/\ell, \quad \Lambda = \lambda/\ell, \quad T = t/t_*\,, \quad t_* = \ell/U_0$$

The governing equation can be rewritten as

$$H_T + \left[\frac{1}{Ca}\left(\frac{1}{3} H^3 + \Lambda H^2 \right)\left(H_{XXX} - Bo \cos\alpha\, H_X + Bo \sin\alpha \right) \right]_X = 0$$

where

$$B_o = \frac{\rho g \ell^2}{\gamma} \quad \text{and} \quad Ca = \frac{\mu U_0}{\gamma}.$$

- Non-zero α and sufficient shallow flows ($h_x \ll \tan\alpha$) with slip BC.

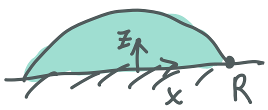
$$H_T + \left[\frac{1}{Ca} \left(\frac{1}{3} H^3 + \Lambda H^2 \right) \left(H_{xxx} + B_0 \sin\alpha \right) \right]_X = 0$$

- Zero α (free surface flow on a horizontal plane) with no-slip BC.

$$H_T + \left[\frac{1}{3Ca} H^3 \left(H_{xxx} - B_0 H_x \right) \right]_X = 0$$

• Spreading: No-slip and slip boundary conditions

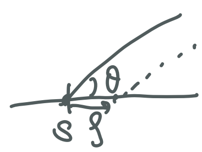
How to model the evolution/spreading of a blob of fluid? May simply apply



$H = 0$, $H_X = -\tan\theta \simeq \theta$ at $X = R$

where R is unknown (to be determined based on mass conservation). However, such b.C.s are not compatible with the governing equation – as we show now.

Suppose that the boundary is located at $x = s(t)$. Zoom in on the interface by setting $x = s(t) + \epsilon$ with ϵ small. Locally we have $h \sim \theta\epsilon$. Set



$$X = S(t) + \xi \quad \text{and} \quad H = \theta\xi + f(\xi) \quad |\xi| \ll 1, \ |f| \ll |\theta\xi| .$$

Now the governing equation reads (with $\frac{d\xi}{dT} = -\dot{S}$, $\frac{\partial \xi}{\partial x} = 1$, $H_x = \theta + f_\xi$)

$$f_T - \underbrace{f_\xi \dot{S}}_{f/\xi} - \theta \dot{S} + \left[\frac{1}{3Ca} \underbrace{\theta^3 \xi^3 \left(f_{\xi\xi\xi}\right.}_{\sim f/\xi} - \underbrace{Bo\,\theta}_{\sim \xi^2} - \left.\underbrace{Bo f_\xi}_{\sim \xi f} \right) \right]_\xi = 0$$

- Attempt to neglect the higher spatial derivatives

$$-f_\xi \dot{S} = \theta \dot{S} \quad \text{with } f=0 \text{ at } \xi = 0$$

$$\rightarrow f = -\theta \xi \quad (\text{Not compatible with } |f| \ll |\theta \xi|) \quad \times$$

- To include higher spatial derivatives

$$\left(\frac{1}{3Ca} \theta^3 \xi^3 f_{\xi\xi\xi} \right)_\xi \sim \theta \dot{S}$$

$$\rightarrow f_{\xi\xi\xi} \sim \frac{3Ca\,\dot{S}}{\theta^2 \xi^2}$$

$$\rightarrow f \sim \frac{3Ca\,\dot{S}}{\theta^2} \left(\xi \log \xi - \cancelto{0}{\xi} \right) \not\ll \theta \xi \quad \times$$

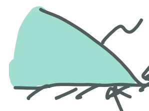
Incompatible again

Check here?

- To include gravity term

$$\rightarrow f_\xi \sim - \frac{\theta \dot{S} \xi \times 3Ca}{Bo\, \theta^3 \xi^3} = - \frac{3Ca\, \dot{S}}{Bo\, \theta^2 \xi^2} \quad (\text{More singular}) \quad \times$$

CANNOT find a consitent solution local to the moving contact line! The difficulty does not lie in the use of thin-film equation or inadequate mathematics — it is a fundemental physical difficulty



An hypothesis for the physics required to treat the moving contact line is to allow a small amount of slip between the substrate and the fluid: $u = \lambda \frac{\partial u}{\partial z}\Big|_{z=0}$.

The thin film equation becomes

$$H_T + \left[\frac{1}{Ca} \left(\frac{1}{3} H^3 + \Lambda H^2 \right) \left(H_{xxx} - B_0 H_x \right) \right]_X = 0 .$$

Use $X = S + \xi$ and $H = \theta \xi + f$ so that

$$f_T - \theta \dot{S} - f_\xi \dot{S} + \left[\frac{1}{Ca} \left(\frac{1}{3} \theta^3 \xi^3 + \Lambda \theta^2 \xi^2 \right) \left(f_{\xi\xi\xi} - B_0 f_\xi - B_0 \theta \right) \right]_\xi = 0$$

The leading-order balance gives

$$\Lambda \theta^2 \xi^2 f_{\xi\xi\xi} \sim Ca\, \theta \dot{S}\, \xi$$

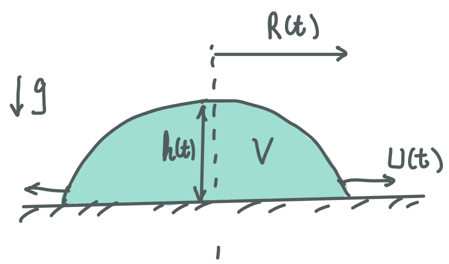
$$\rightarrow \quad f_{\xi\xi} \sim \frac{Ca\, \dot{S}}{\theta \Lambda} \log \xi$$

$$\rightarrow \quad f \sim \frac{Ca\, \dot{S}}{\theta \Lambda} \xi^2 \log \xi \ll \theta \xi \quad \checkmark$$

So that it is possible to impose a contact angle θ with slip bcs.

• Precursor film and Tanner's law

Let us first check how a drop spreads via a scaling argument.



• $Bo = \frac{\rho g R^2}{\gamma} \gg 1$, $Re \ll 1$

Driven by gravity, resisted by viscosity

$$\frac{\partial p}{\partial r} = \mu \frac{\partial^2 u}{\partial z^2} \rightarrow \frac{\rho g h}{R} \sim \mu \frac{U}{h^2} \sim \mu \frac{R}{h^2 t}$$

Continuity $\pi R^2 h \sim V \rightarrow h \sim \frac{V}{R^2}$

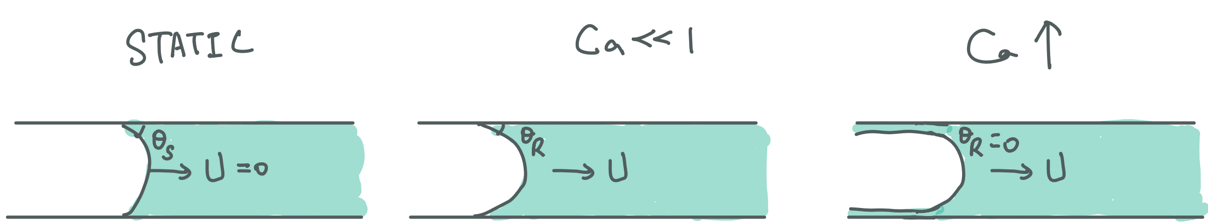
$$\Rightarrow \boxed{R \sim \left(\frac{\rho g V^3}{\mu}\right)^{1/8} t^{1/8}}$$

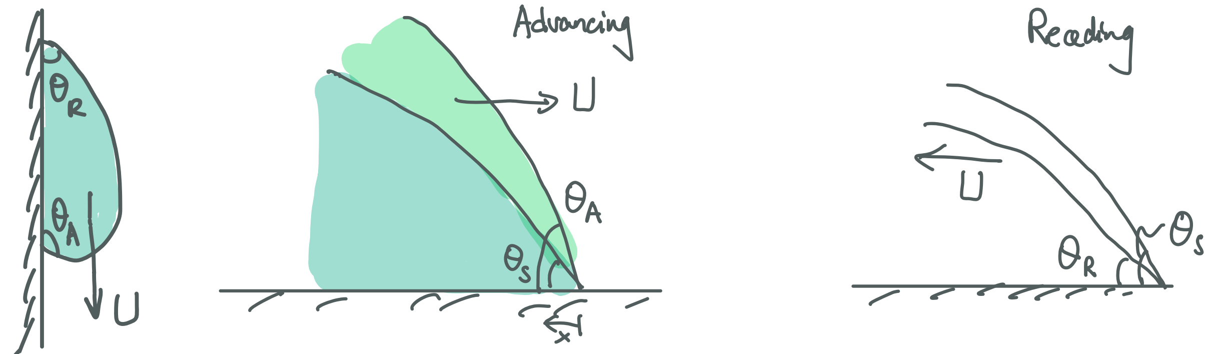
• $Bo = \frac{\rho g R^2}{\gamma} \ll 1$, $Re \ll 1$ (Note that gravity becomes progressively more important as $R \uparrow$)

$$\gamma \frac{h}{R^3} \sim \mu \frac{U}{h^2} \sim \mu \frac{R}{h^2 t} \Rightarrow \gamma \frac{h^3}{R^4} \sim \frac{\mu}{t} \Rightarrow \boxed{R \sim \left(\frac{\gamma V^3}{\mu}\right)^{1/10} t^{1/10}}$$

How to figure out the prefactor? To be discussed in next lecture.

(1) Advancing and receding angle (A regularization mechanism like slip)





$\theta_A > \theta_S > \theta_R$ in general.

For the spreading problem, let's see what is going on near the advancing front.

- Geometry. For $\theta_S < \theta_A \ll 1$, $\tan\theta_A \simeq \frac{h}{x} \simeq \theta_A$, $h \simeq \theta_A x$

- Velocity gradient. $\frac{\partial u}{\partial z} \sim \frac{U}{h} = \frac{U}{\theta_A x}$

- Force balance. Driving force $F = \gamma_{SV} - \gamma_{SL} - \gamma\cos\theta_A$

$$= \gamma(\cos\theta_S - \cos\theta_A) > 0 \quad \text{since } \theta_S < \theta_A$$

- Revisiting the scaling arguement

$$\underbrace{\mu\frac{U}{h}}_{\text{Viscous stress}} \times \pi R^2 \sim \underbrace{\gamma(\cos\theta_S - \cos\theta_A)}_{\text{Line force}} \times 2\pi R$$

$$\Rightarrow \mu\frac{dR}{dt}\times\frac{R^2}{V}\times R^2 \sim \gamma(\cos\theta_S - \cos\theta_A)R \rightarrow \boxed{R \sim \left(\frac{\gamma V}{\mu}\right)^{1/4} t^{1/4}}$$

Not what has been observed. why? $\theta_A \sim f(U)$

② Physical picture of precusor film (complete wetting)

First, relate the driving force F to the local velocity by

$$FU \sim \phi$$

where ϕ is viscous dissipation in the corner. Locally, we have

$$\phi \sim \int_A \mu \left(\frac{\partial u}{\partial z}\right)^2 dA \sim \int_0^\infty dx \int_0^{h=\theta_A x} \mu \left(\frac{U^2}{\theta_A x}\right) dz$$

$$\sim \frac{\mu U^2}{\theta_A} \int_a^R \frac{1}{x}\, dx$$

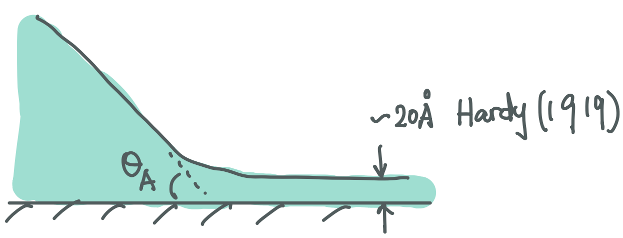
R ← Drop size

a ← Molecular size

$$\phi \sim \frac{3\mu U^2 l_D}{\theta_A}, \quad l_D \sim \int_a^R \frac{1}{x}\, dx = \ln R/a$$

De Gennes' approximation, $15 < l_D < 20$ in experiments

Then calculate F assuming a thin film covering the substrate (in the presence of precursor film).



$$F = \gamma + \gamma_{sl} - \gamma\cos\theta_A - \gamma_{sl}$$

$$\simeq \frac{1}{2}\gamma\theta_A^2$$

(May use this to revisit the scaling above and obtain the same result)

$\mu \frac{U}{h} R^2 \sim \gamma \theta_A^2 R$ & $h \sim R\theta_A$

$$FU \sim \phi \rightarrow \frac{1}{2}\gamma\theta_A^2 U \sim \frac{3\mu U^2 l_D}{\theta_A} \rightarrow U \sim \frac{\gamma l_D}{\mu}\theta_A^3 \rightarrow \theta_A^3 \propto Ca \text{ (Tanner's law)}.$$

Finally, use the conservation of mass $V \sim R^3\theta_A$ to obtain

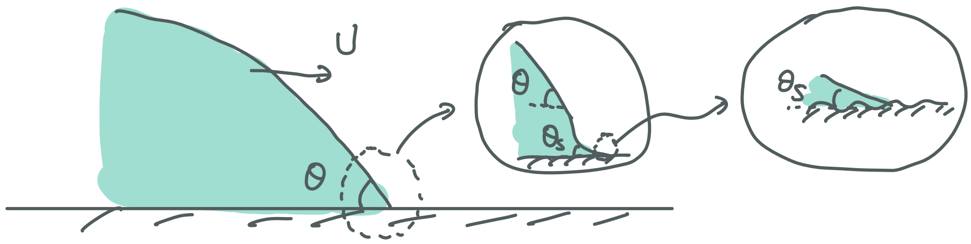
$$0 = \theta_A R^2 \frac{dR}{dt} + \frac{d\theta_A}{dt} \cdot R^3$$

$$\frac{1}{\theta_A}\frac{d\theta_A}{dt} \sim -\frac{1}{R}\underbrace{\frac{dR}{dt}}_{U} \sim -\frac{\gamma}{\mu R}\theta_A^3 \sim -\frac{\gamma}{\mu V^{1/3}}\theta_A^{\frac{10}{3}}$$

$$\rightarrow \theta_A \sim \left(\frac{\mu V^{1/3}}{\gamma t}\right)^{3/10}, \quad R \sim \left(\frac{V}{\theta_A}\right)^{1/3} \sim \left(\frac{\gamma V^3}{\mu}\right)^{1/10} t^{1/10}$$

(consistent with the thin-film behavior)

③ Cox-Voinov law

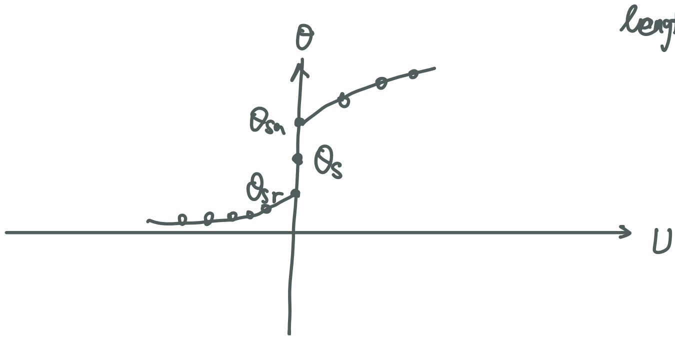


Matching of solutions at different scales gives rise to

$$|U| = \begin{cases} K(\theta^3 - \theta_{sa}^3) \\ -K(\theta^3 - \theta_{sr}^3) \end{cases}, \quad K = \frac{\gamma}{9\mu \ln(\alpha l_0 / l_i)}$$

Depends on details of micro/macro solutions

Outer length Inner length.



In summary, the "singularity" of moving contact line can be "regularized" by "slip", "precursor film", "C-V law" and other multiscale modelling.

· Quasistatic evolution with $Ca \ll 1$, $Bo \ll 1$

Using CV law in full-time dependent problem often still requirs some slip to avoid a contact line singularity. However, there is a natural velocity scale in CV law and if it is slow ($Ca = \frac{\mu k}{\gamma} \ll 1$), the fluid moves quasi-statically.

We the solve the static version of

$$H_T + \left[\frac{1}{Ca} \left(\frac{1}{3} H^3 + \Lambda H^2 \right) (H_{xxx} - B_0 H_x) \right]_X = 0 .$$

The contact line moves with a speed determined by the contact angle. We don't have to include slip in this case (no stress singularity). Let us focus on a 2D example

$$Ca = \frac{\mu K}{\gamma} \ll 1 , \quad B_0 \ll 1 , \quad U_0 = K , \quad l = A^{1/2}$$

The thin film equation is now

$$\left(H^3 H_{xxx} \right)_X = 0$$

subject to

$$H_x(0) = 0$$

and

$$H_x(S) = -\theta$$

$$H(S) = 0$$

$$\dot{S} = \theta^3 - \theta_{SA}^3$$

at the contact line $X = S(t)$, together with a global mass balance equation

$$\int_0^S H \, dX = \frac{1}{2}$$

We find that

$$H = \frac{\theta}{2S}\left(S^2 - X^2\right)$$

Imposing the total mass constraint gives

$$\frac{2}{3}\theta S^2 = 1$$

Using GV law gives the evolution equation S as

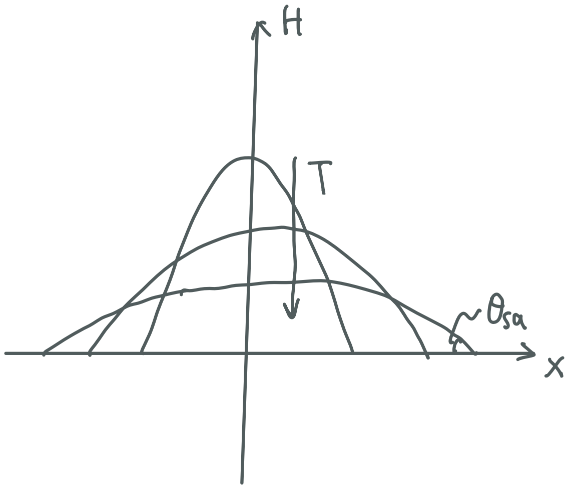
$$\frac{dS}{dT} = \frac{27}{8}\frac{1}{S^6} - \theta_{sa}.$$

This is an ODE for $S(t)$ with $S(0) = S_0$.

- For $\theta_{sa} \ll 1$, $S^7 - S_0^7 \sim T$

$$\rightarrow \frac{R}{A^{1/2}} \sim \left(\frac{t}{A^{1/2}/(\gamma/\mu)}\right)^{1/7}$$

$$R \sim \left(\frac{\gamma A^3}{\mu}\right)^{1/7} t^{1/7} \quad \text{for 2D case.}$$



- Eventually, $S \rightarrow S_\infty$ as $T \rightarrow \infty$ when $H_x = -\theta_{sa}$.

(Note that for the axisymmetric case, the mass conservation becomes $\theta S^3 \sim 1$, which will lead $\frac{dS}{dt} \sim \frac{1}{S^9}$ or $S^{10} - S_0^{10} \sim T$)