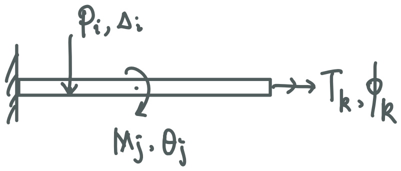
第八章 弹性杆系结构的一般性质

§8.1. Lagrang定理 & Castigliano定理

任意弹性系统



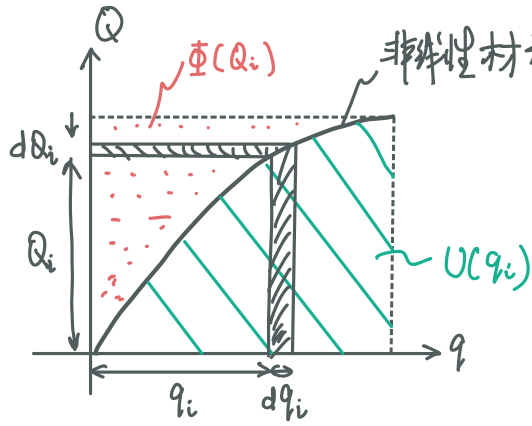
• $U = U(\underbrace{\Delta_i, \theta_j, \phi_k, \cdots}_{q})$ 弹性能 ▧ 区域：广义位移的函数

• $\Phi = \Phi(\underbrace{P_i, M_j, T_k, \cdots}_{Q})$ 余能 ⊡ 区域：广义力的函数

广义力 Q：P_i, M_j, $T_k \cdots$

广义位移 q：Δ_i, θ_j, $\phi_k \cdots$

一维示意图



• 考查在广义位移 $\overbrace{\delta q}^{\text{虚位移}}$ 微扰下，系统弹性能的变化（有两种方式）

① $\delta W = \sum_i P_i \delta\Delta_i + \sum_j M_j \delta\theta_j + \sum_k T_k \delta\phi_k + \cdots = \sum_i Q_i \delta q_i$ （物理上）

② $\delta U = \sum_i \frac{\partial U}{\partial \Delta_i}\delta\Delta_i + \sum_j \frac{\partial U}{\partial \theta_j}\delta\theta_j + \sum_k \frac{\partial U}{\partial \phi_k}\delta\phi_k + \cdots = \sum_i \frac{\partial U}{\partial q_i}\delta q_i$ （函数性质 as $\delta q \to 0$）

虚位移 δq_i 为任意小量

且 $\underbrace{\delta U}_{\text{内虚功}} = \underbrace{\delta W}_{\text{外虚功}}$ $\rightarrow \begin{cases} Q_i = \dfrac{\partial U}{\partial q_i} \\ \delta U = \sum_i Q_i \delta q_i \end{cases}$ $\xrightarrow{\text{或}}$

$$\boxed{\begin{aligned} &U = U(\Delta_i, \theta_j, T_k, \cdots) \\ &P_i = \frac{\partial U}{\partial \Delta_i} \\ &M_j = \frac{\partial U}{\partial \theta_j} \\ &T_k = \frac{\partial U}{\partial \phi_k} \end{aligned}}$$

Lagrang定理

· 考查任意 δQ 微扰下余能 $\Phi=\Phi(Q)$ 的变化（同样两种考查方式）

① 数学上，$\delta\Phi=\sum_i \frac{\partial\Phi}{\partial Q_i}\delta Q_i$

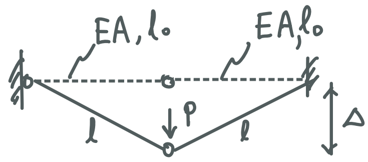
② 物理上，$\Phi=\sum_i Q_i q_i - U(q_i) \to \delta\Phi=\sum_i(\delta Q_i q_i + \cancel{Q_i\delta q_i}) - \overbrace{\cancel{\sum_i Q_i\delta q_i}}^{\delta U}$

$$\delta Q_i\text{任意} \to \begin{cases} q_i=\dfrac{\partial\Phi}{\partial Q_i} \\ \text{(1-维示意图上比较显然)} \\ \delta\Phi=\sum_i q_i\,\delta Q_i \end{cases} \xrightarrow{\text{或}} \boxed{\begin{aligned} &\Phi=\Phi(P_i, M_j, T_k,\cdots) \\ &\Delta_i=\frac{\partial\Phi}{\partial P_i} \\ &\theta_j=\frac{\partial\Phi}{\partial M_j} \\ &\phi_k=\frac{\partial\Phi}{\partial T_k} \end{aligned}}$$

Castigliano 定理

值得一提的是 拉氏、卡氏定理推导并没有做线性材料、几何假设，因此可用于解答非线性问题。

例：几何非线性



P vs. Δ?

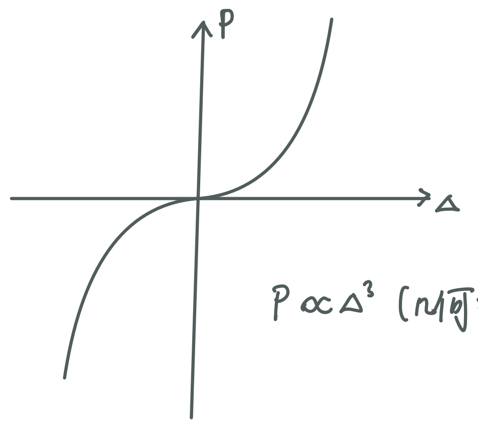
非线性来自于几何，因此采用几何 q 描述 U，进而采用 Lagrange 定理

$$l=\sqrt{l_0^2+\Delta^2}\simeq l_0\left(1+\frac{1}{2}\frac{\Delta^2}{l_0^2}\right)$$

$$\varepsilon=\frac{l-l_0}{l_0}=\frac{1}{2}\frac{\Delta^2}{l_0^2}$$

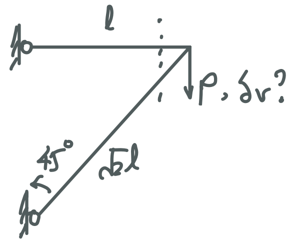
$$U=2\times\frac{1}{2}E\varepsilon^2 A l_0=\frac{1}{4}EA\frac{\Delta^4}{l_0^3}$$

$$P=\frac{\partial U}{\partial\Delta}=EA\frac{\Delta^3}{l_0^3}$$



$P\propto\Delta^3$（几何非线性）

例：材料非线性



对于线弹性结构，我们有 $U=\frac{N^2 l}{2EA}$，

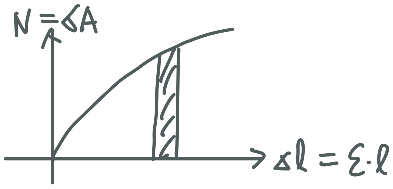
$$\phi = N\Delta l - U = N\frac{Nl}{EA} - U = U$$

材料关系 $\sigma = B\varepsilon^{1/2}$

(认为拉压对称)

对于非线性结构，$dU = d(\Delta l)\cdot N = l\,d\varepsilon \cdot AB\varepsilon^{1/2}$

$$U(\varepsilon)=\int_0^{\varepsilon} AB\varepsilon^{1/2} l\,d\varepsilon = \frac{2}{3}ABl\cdot\varepsilon^{\frac{3}{2}}$$



$$\phi(N) = N\cdot\Delta l - U = N\varepsilon l - \frac{2}{3}ABl\varepsilon^{\frac{3}{2}}$$

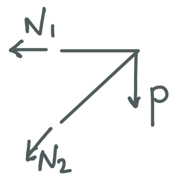
$$= \frac{N^3 l}{A^2B^2} - \frac{2}{3}ABl\cdot\frac{N^3}{A^3B^3} = \frac{1}{3}\frac{N^3 l}{A^2B^2}$$

$N = AB\varepsilon^{1/2}$

$\varepsilon = \left(\frac{N}{AB}\right)^2$

也可以直接 $\phi(N)=\int_0^N \Delta l\,dN = \int_0^N \left(\frac{N}{AB}\right)^2 l\,dN = \frac{1}{3}\frac{N^3 l}{A^2B^2}$

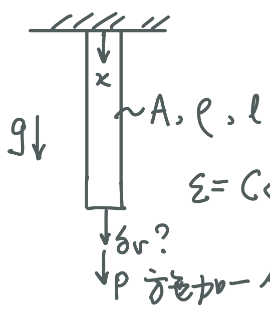
平衡关系：



$$\begin{aligned} N_1+\frac{\sqrt{2}}{2}N_2&=0 \\ \frac{\sqrt{2}}{2}N_2+P&=0 \end{aligned} \rightarrow \begin{cases} N_1 = P \\ N_2 = -\sqrt{2}P \end{cases}$$

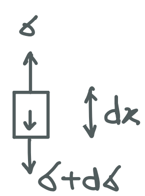
$$\rightarrow \phi = \frac{P^3 l}{3A^2B^2} + \frac{2\sqrt{2}P^3\sqrt{2}l}{3A^2B^2} = \frac{5}{3}\frac{P^3 l}{A^2B^2}, \quad \delta v = \frac{\partial \phi}{\partial P} = \frac{5P^2 l}{A^2B^2} \quad (\text{卡氏定理})$$

例：非均匀变形



$\varepsilon = C\sigma^n$

施加一个虚拟的集中力



$$d\sigma\cdot A + \rho g\,dx\cdot A = 0$$

$$\rightarrow \frac{d\sigma}{dx} = -\rho g \rightarrow \sigma = \rho g(l-x) + \frac{P}{A}$$

微元 dx 的余能 $d\phi = \int_0^{\sigma(x)} \varepsilon\, d\sigma \cdot (A dx) = A dx \int_0^{\sigma} C\sigma^n d\sigma = \frac{CA}{n+1}\sigma^{n+1} dx$

杆的余能 $\phi = \int_0^l d\phi = \int_0^l \frac{CA}{n+1}\left[\rho g(l-x) + \frac{P}{A}\right]^{n+1} dx$

$$\delta_v = \frac{\partial\phi}{\partial P}\Big|_{P=0} = \int_0^l CA\left[\rho g(l-x) + \frac{P}{A}\right]^n \cdot \frac{1}{A}\Big|_{P=0} dx$$

$$= \int_0^l c(\rho g)^n (l-x)^n\, dx$$

$$= \frac{-c(\rho g)^n}{n+1}(l-x)^{n+1}\Big|_0^l = \frac{c(\rho g)^n l^{n+1}}{n+1}$$

量纲检查：

$C \sim \frac{L^{2n}}{F^n}$，$\rho g \sim \frac{F}{L^3}$

$\delta \sim \frac{L^{2n}}{F^n} \cdot \frac{F^n}{L^{3n}} L^{n+1} \sim L$ ✓

§8.2. 线性弹性结构

对于线弹性结构，平衡方程可以在系统未变形状态下建立，变形与位移呈线性关系，材料服从（广义）胡克定律.

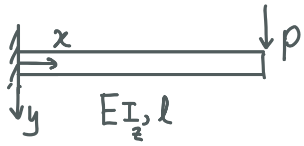
$$U(q_i) = \frac{1}{2}Q_i q_i = Q_i q_i - \frac{1}{2}Q_i q_i = \phi(Q_i)$$

具体而言，拉压弯扭变形下的变形能为：

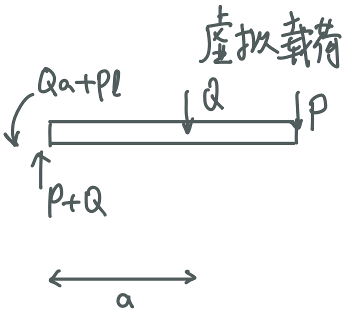
$$\phi = \sum \frac{N^2 l}{2EA} + \sum \int_0^x \frac{M_x^2}{2GI_p} dx + \sum \int_0^x \frac{M_z^2}{2EI_z} dx$$

$$U = \sum \frac{1}{2}EAl\underset{\uparrow\, \frac{du}{dx} = \frac{\Delta l}{l}}{\varepsilon^2} + \sum \int_0^x \frac{1}{2}GI_p \underset{\uparrow\, \frac{d\varphi}{dx}}{\theta}\, dx + \sum \int_0^x \frac{1}{2}EI_z \underset{\uparrow\, \frac{d^2 v}{dx^2}}{\kappa^2}\, dx$$

例：



求 $v = v(x)$



$$M = (Qa+Pl)\varphi_0(x-0) - (P+Q)\varphi_1(x-0) + Q\varphi_1(x-a)$$

$$= Q(a-x) + P(l-x) + Q\varphi_1(x-a)$$

$$v(a) = \left.\frac{\partial \Phi}{\partial Q}\right|_{Q=0}$$

$$= \frac{1}{2EI_z}\int_0^l \frac{\partial}{\partial Q}\left[Q(a-x) + P(l-x) + Q\varphi_1(x-a)\right]^2\Big|_{Q=0} dx$$

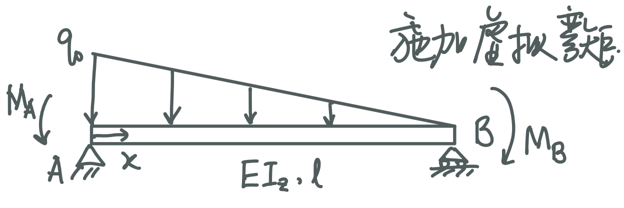
$$= \frac{1}{2EI_z}\int_0^l 2\left[Q(a-x) + P(l-x) + Q\varphi_1(x-a)\right]\left[(a-x) + \varphi_1(x-a)\right]\Big|_{Q=0} dx$$

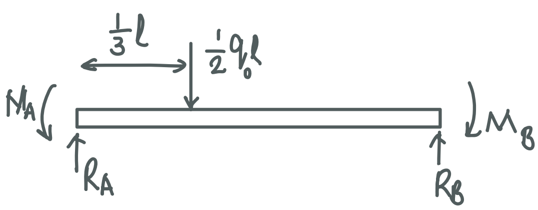
$$= \frac{1}{2EI_z}\left\{\int_0^a 2P(l-x)(a-x)\,dx\right.$$

$$= \frac{P}{EI_z}\left(+\frac{1}{3}x^3 - \frac{1}{2}(a+l)x^2 + alx\right)\Big|_0^a = \frac{P}{EI_z}\left(-\frac{1}{6}a^3 + \frac{1}{2}la^2\right)$$

$$\rightarrow v(x) = \frac{P}{6EI_z}\left(-x^3 + 3lx^2\right)$$

例： 求 A、B 处转角



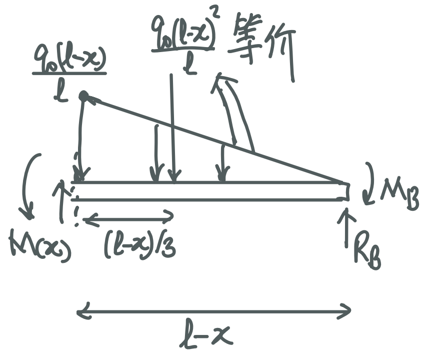


平衡：$\Sigma F_y = R_A + R_B - \frac{1}{2}ql = 0$

$$\Sigma M_z^A = \frac{1}{6}q_0 l^2 - R_B l + M_B - M_A = 0$$

$$\rightarrow R_B = \frac{1}{6}q_0 l + \frac{M_B - M_A}{l}$$

$$R_A = \frac{1}{3}q_0 l + \frac{M_A - M_B}{l}$$



$$M(x) = M_B - R_B\cdot(l-x) + \frac{1}{2}\frac{q_0(l-x)}{l}\cdot(l-x)\times\frac{1}{3}(l-x)$$

$$= M_B - \frac{1}{6}q_0 l(l-x) - \frac{M_B-M_A}{l}\cdot(l-x) + \frac{1}{6}\frac{q_0}{l}(l-x)^3$$

Check: $M(0) = M_B - \frac{1}{6}q_0 l^2 - M_B + M_A + \frac{1}{6}q_0 l^2 = M_A$ ✓

$$\phi = \frac{1}{2EI_z}\int_0^l M^2\,dx$$

$$\theta_A = \frac{\partial\phi}{\partial M_A}\Big|_{M_A=M_B=0}$$

$$= \frac{1}{EI_z}\int_0^l\left[M_B - \frac{1}{6}q_0 l(l-x) - \frac{M_B-M_A}{l}(l-x) + \frac{1}{6}\frac{q_0}{l}(l-x)^3\right]\cdot\frac{(l-x)}{l}\,dx$$

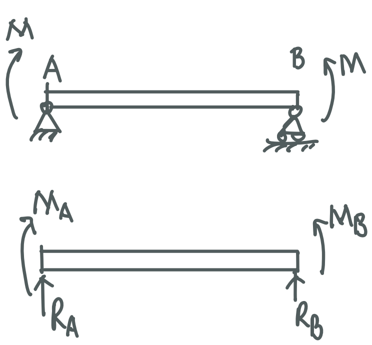
$$= \frac{1}{EI_z}\int_0^l\left[-\frac{1}{6}q_0(l-x)^2 + \frac{1}{6}\frac{q_0}{l^2}(l-x)^4\right]dx$$

$$= \frac{q_0}{EI_z}\left[\frac{1}{18}(l-x)^3 - \frac{1}{30}\frac{1}{l^2}(l-x)^5\right]\Big|_0^l = -\frac{1}{45}\frac{q_0 l^3}{EI_z}$$ （负号说明转动方向与 M_A 方向相反）

$$\theta_B = \frac{\partial\phi}{\partial M_B}\Big|_{M_A=M_B=0} = \frac{1}{EI_z}\int_0^l\left[M_B - \frac{1}{6}q_0 l(l-x) - \frac{M_B-M_A}{l}(l-x) + \frac{1}{6}\frac{q_0}{l}(l-x)^3\right]\cdot\left[1-\frac{l-x}{l}\right]dx$$

$$= -\frac{7}{360}\frac{q_0 l^3}{EI_z}$$ （负号说明转动方向与 M_B 方向相反）

例：求A端转角



$R_A = \frac{M_B-M_A}{l}$, $M(x) = -M_A - \frac{M_B-M_A}{l}x$.

$$\theta_A = \frac{\partial\phi}{\partial M_A}\Big|_{M_A=M_B=M}$$

教材有Typo

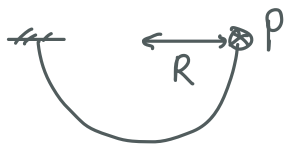
$$= \int_0^l\frac{1}{EI_z}\left(-M_A - \frac{M_B-M_A}{l}x\right)\cdot\left(-1+\frac{x}{l}\right)dx = \frac{M}{2EI_z}$$

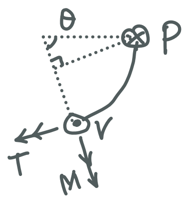
或 $\delta W = M_A\delta\theta_A + M_B\delta\theta_B = 2M\delta\theta = Q_i\delta q_i$

令 $Q = M$, $q = 2\theta$

则 $q = \frac{\partial\Phi}{\partial M} = \partial\left(\frac{M^2 l}{2EI_z}\right)/\partial M = \frac{Ml}{EI_z} \rightarrow \theta = \frac{Ml}{EI_z}$

例：弯扭问题





$V = P$

$T = PR(1-\cos\theta)$

$M = PR\sin\theta$

$$\Delta = \frac{\partial \phi}{\partial P} = \int_0^{\pi}\left(\frac{M}{EI_z}\frac{\partial M}{\partial P} + \frac{T}{GI_p}\frac{\partial T}{\partial P}\right) R\,d\theta$$

忽略剪力对应的变形能

$$= \int_0^{\pi}\left[\frac{PR^2\sin^2\theta}{EI_z} + \frac{PR^2(1-\cos\theta)^2}{GI_p}\right] R\,d\theta$$

$$= \frac{PR^2}{EI_z}\cdot\frac{\pi}{2} + \frac{PR^2}{GI_p}\frac{3\pi}{2}$$

圆形截面 $I_z = \frac{\pi a^4}{4}$, $I_p = \frac{\pi a^4}{2}$

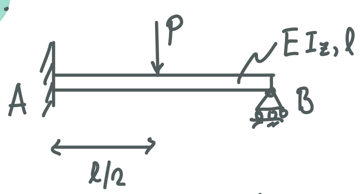
$G = \frac{E}{2(1+\nu)}$

$$= \frac{PR^2}{Ea^4}\left(2 + 6(1+\nu)\right) = \frac{2PR^2}{Ea^4}(4+3\nu)$$

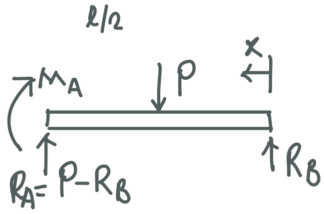
§8.3. 线弹性静不定问题

对于 n 次静不定杆系，存在 Q_i 以及 n 个约束反力，通过 n 个约束条件求解.

例：



求约束反力



$M_A = R_B l - \frac{1}{2}Pl.$

建立以B为原点的坐标系方便计算.

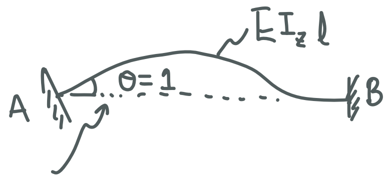
$$M(x) = -R_B\varphi_1(x-0) + P\varphi_1\left(x-\frac{l}{2}\right)$$

$$v_B = \frac{\partial \phi}{\partial R_B} = \int_0^l \frac{M}{EI_z}\cdot\frac{\partial M}{\partial R_B}\,dx = \frac{1}{EI_z}\int_0^l \left[R_B x + P\varphi_1\left(x-\frac{l}{2}\right)\right]\cdot(-x)\,dx$$

$$\to EI_z v_B = \int_0^{\frac{l}{2}} R_B x^2\,dx + \int_{\frac{l}{2}}^{l} \left[+R_B x^2 - Px\left(x-\frac{l}{2}\right)\right] dx$$

$$= \frac{1}{24}R_B l^3 + \underbrace{\left(\frac{1}{3}R_B x^3 - \frac{1}{3}Px^3 + \frac{1}{4}Plx^2\right)\Bigg|_{l/2}^{l}}_{\frac{1}{3}R_B l^3 - \frac{1}{24}R_B l^3 - \frac{7}{24}Pl^3 + \frac{3}{16}Pl^3} = 0 \quad \to \quad R_B = \frac{5}{16}P,\ R_A = \frac{11}{16}P,\ M = -\frac{3}{16}Pl$$

例:



强迫以转角 $\theta=1$ 且 $v=0$

求支反力?

$$M(x) = M_A - R_A x$$

$$\phi = \frac{1}{2}\int_0^l \frac{M^2}{EI_z}dx = \frac{1}{2EI_z}\int_0^l \left(M_A^2 - 2R_A M_A x + R_A^2 x^2\right)dx$$

$$= \frac{1}{2EI_z}\left(M_A^2 l - R_A M_A l^2 + \frac{1}{3}R_A^2 l^3\right)$$

几何/协调条件:

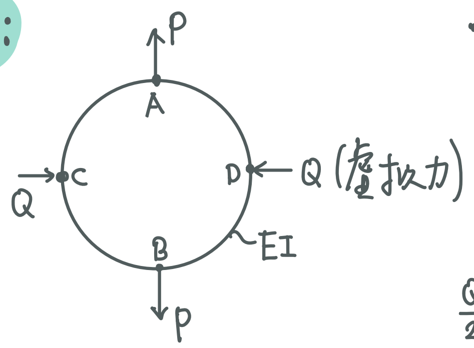
$$v_A = \frac{\partial\phi}{\partial R_A} = \frac{1}{2EI_z}\left(-M_A l^2 + \frac{2}{3}R_A l^3\right) = 0 \to M_A = \frac{2}{3}R_A l$$

$$\theta_A = \frac{\partial\phi}{\partial M_A} = \frac{1}{2EI_z}\left(2M_A l - R_A l^2\right) = 1 \to R_A = \frac{6EI_z}{l^2} = R_B$$

$$M_A = \frac{4EI_z}{l}$$

$$M_B = -\frac{2EI_z}{l^2}$$

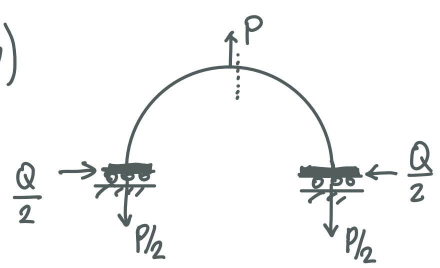
例:

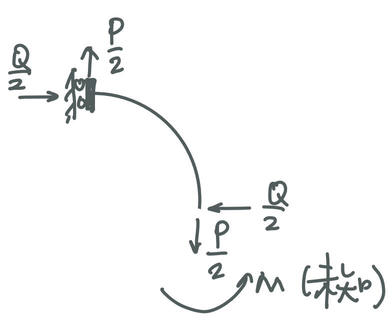


求AB两点发生的位移

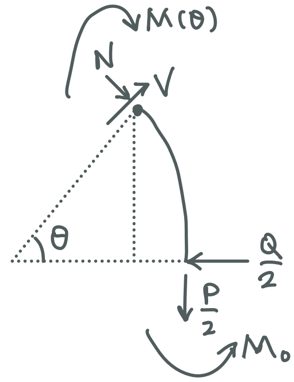
CD两点发生的位移

· A, B, C, D点无转动 ($\theta=0$), 无剪力 ($v'''=0$)





注意若无Q, 剪力为0 (对称性 why?)



忽略 N, V 对应的拉压、剪切变形能.

$$M(\theta) = M_0 - \frac{1}{2}QR\sin\theta - \frac{P}{2}R(1-\cos\theta)$$

$$\phi = \int_0^{\pi} \frac{1}{2EI}\left[M_0 - \frac{1}{2}QR\sin\theta - \frac{1}{2}PR(1-\cos\theta)\right]^2 R\,d\theta$$

↑ 1/4 的变形能

首先确定 M_0.
$$\theta_0 = \frac{\partial\phi}{\partial M_0} = \frac{R}{EI}\int_0^{\frac{\pi}{2}}\left[M_0 - \frac{1}{2}QR\sin\theta - \frac{1}{2}PR(1-\cos\theta)\right]\cdot(1)\,d\theta = 0$$

$$\frac{\pi}{2}M_0 - \frac{1}{2}QR\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta - \frac{1}{2}PR\int_0^{\frac{\pi}{2}}(1-\cos\theta)\,d\theta = 0$$

$$\rightarrow M_0 = \frac{1}{\pi}QR + \frac{1}{\pi}PR\left(\frac{\pi}{2}-1\right)$$

$$M(\theta) = QR\left(\frac{1}{\pi} - \frac{1}{2}\sin\theta\right) + PR\left(-\frac{1}{\pi} + \frac{1}{2}\cos\theta\right)$$

δ_{AB} 伸长量
$$\delta_{AB} = 4\times\frac{\partial\phi}{\partial P}\bigg|_{Q=0} = 4\int_0^{\frac{\pi}{2}}\frac{1}{EI}\cdot PR\left(-\frac{1}{\pi}+\frac{1}{2}\cos\theta\right)\underbrace{R\left(-\frac{1}{\pi}+\frac{1}{2}\cos\theta\right)}_{\frac{\partial M}{\partial P}}R\,d\theta$$

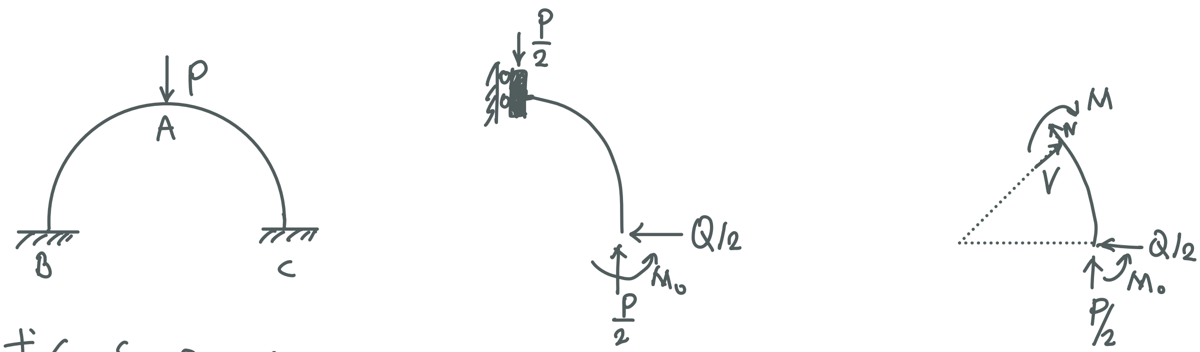
$$= \frac{4PR^3}{EI}\underbrace{\int_0^{\frac{\pi}{2}}\left(\frac{1}{\pi^2} - \frac{1}{\pi}\cos\theta + \frac{1}{4}\cos^2\theta\right)d\theta}_{\left(\frac{1}{2\pi} - \frac{1}{\pi} + \frac{1}{4}\cdot\frac{\pi}{4}\right)} = \frac{PR^3}{EI}\underbrace{\left(\frac{\pi}{4} - \frac{2}{\pi}\right)}_{\doteq 0.149}$$

δ_{CD} 缩短量
$$\delta_{CD} = 4\times\frac{\partial\phi}{\partial Q}\bigg|_{Q=0} = 4\int_0^{\frac{\pi}{2}}\frac{1}{EI}PR\left(-\frac{1}{\pi}+\frac{1}{2}\cos\theta\right)\cdot R\left(\frac{1}{\pi} - \frac{1}{2}\sin\theta\right)R\,d\theta$$

$$= \frac{4PR^3}{EI}\int_0^{\frac{\pi}{2}}\left(-\frac{1}{\pi^2} + \frac{1}{2\pi}\cos\theta + \frac{1}{2\pi}\sin\theta - \frac{1}{8}\sin2\theta\right)d\theta$$

$$= \frac{4PR^3}{EI}\left(-\frac{1}{2\pi} + \frac{1}{2\pi} + \frac{1}{2\pi} - \frac{1}{8}\right) = \frac{PR^3}{EI}\underbrace{\left(\frac{2}{\pi} - \frac{1}{2}\right)}_{\doteq 0.137}$$

例:



求 δ_v, $\underbrace{\delta_H, \theta}_{=0\ (\text{对称性})}$ at A.

注意这种情况下 $Q\neq 0$

由上一例题可知: $M(\theta) = M_0 - \frac{1}{2}QR\sin\theta + \overset{P\to -P}{\frac{P}{2}}R(1-\cos\theta)$

$$\theta_c = \frac{\partial\phi}{\partial M_0}=0 \to M_0 = \frac{1}{\pi}QR - \frac{1}{\pi}PR\left(\frac{\pi}{2}-1\right)$$

$$M(\theta) = QR\left(\frac{1}{\pi} - \frac{1}{2}\sin\theta\right) - PR\left(-\frac{1}{\pi}+\frac{1}{2}\cos\theta\right)$$

确定支反力 $\delta_H^c = \frac{\partial\phi}{\partial Q} = 0$

$$\to \int_0^{\frac{\pi}{2}}\frac{1}{EI}\left[QR\left(\frac{1}{\pi}-\frac{1}{2}\sin\theta\right) - PR\left(-\frac{1}{\pi}+\frac{1}{2}\cos\theta\right)\right]\overbrace{R\left(\frac{1}{\pi}-\frac{1}{2}\sin\theta\right)}^{\frac{\partial M}{\partial Q}}R\,d\theta=0$$

$$Q\int_0^{\frac{\pi}{2}}\left(\frac{1}{\pi^2}-\frac{1}{\pi}\sin\theta+\frac{1}{4}\sin^2\theta\right)d\theta - P\int_0^{\frac{\pi}{2}}\left(-\frac{1}{\pi^2}+\frac{1}{2\pi}\cos\theta+\frac{1}{2\pi}\sin\theta-\frac{1}{8}\sin 2\theta\right)d\theta=0$$

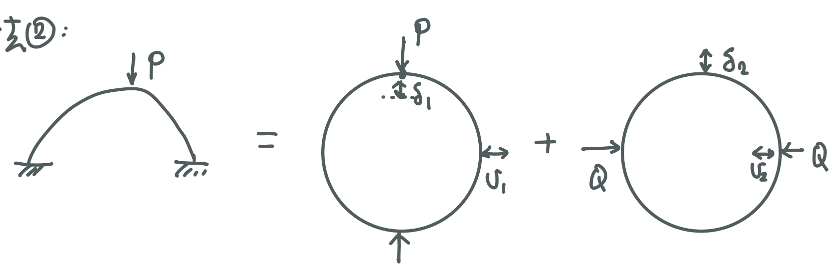
$$\to Q = P\left(\frac{1}{2\pi}-\frac{1}{8}\right)\Big/\left(\frac{\pi}{16}-\frac{1}{2\pi}\right) = P\,\frac{8-2\pi}{\pi^2-8}$$

$$\delta_v^A = 2\times\frac{\partial\phi}{\partial P}\Big|_{M_0,Q} = 2\int_0^{\frac{\pi}{2}}\frac{1}{EI}\left[QR\left(\frac{1}{\pi}-\frac{1}{2}\sin\theta\right) - PR\left(-\frac{1}{\pi}+\frac{1}{2}\cos\theta\right)\right]\left[\frac{8-2\pi}{\pi^2-8}\cdot R\left(\frac{1}{\pi}-\frac{1}{2}\sin\theta\right) - R\left(-\frac{1}{\pi}+\frac{1}{2}\cos\theta\right)\right]R\,d\theta$$

(向下)

$$= \frac{PR^3}{EI}\,\frac{32-20\pi+\pi^3}{8(\pi^2-8)} \simeq 0.0117\,\frac{PR^3}{EI}$$

方法②:

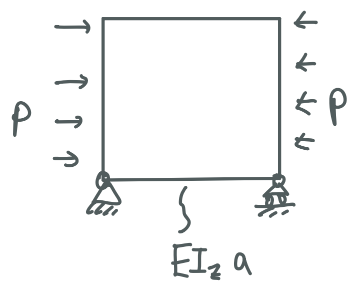


$$\delta_v^A = \delta_1 - \delta_2 = \frac{PR^3}{2EI}\left(\frac{\pi}{4}-\frac{2}{\pi}\right) - \frac{QR^3}{2EI}\left(\frac{2}{\pi}-\frac{1}{2}\right)$$

需施加多少Q? $v_1 = v_2 \rightarrow \frac{PR^3}{2EI}\left(\frac{2}{\pi}-\frac{1}{2}\right) = \frac{QR^3}{2EI}\left(\frac{\pi}{4}-\frac{2}{\pi}\right)$

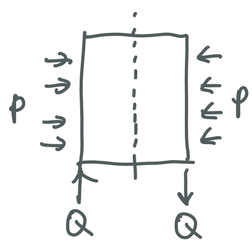
$$\rightarrow \delta_v^A = \frac{PR^3}{2EI}\left[\left(\frac{\pi}{4}-\frac{2}{\pi}\right)-\left(\frac{2}{\pi}-\frac{1}{2}\right)^2\Big/\left(\frac{\pi}{4}-\frac{2}{\pi}\right)\right] = \frac{PR^3}{EI}\,\frac{32-20\pi+\pi^3}{8(\pi^2-8)} \checkmark$$

例: 求弯矩分布(教材P346有Typo, 见勘误表)

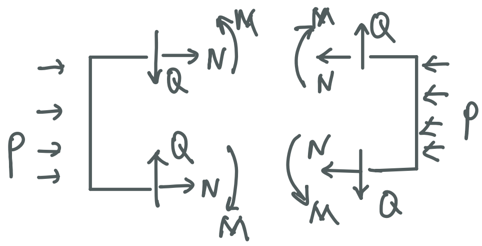


方形刚架

首先尽量利用对称性确定支反力.



左右对称 → $Q=0$

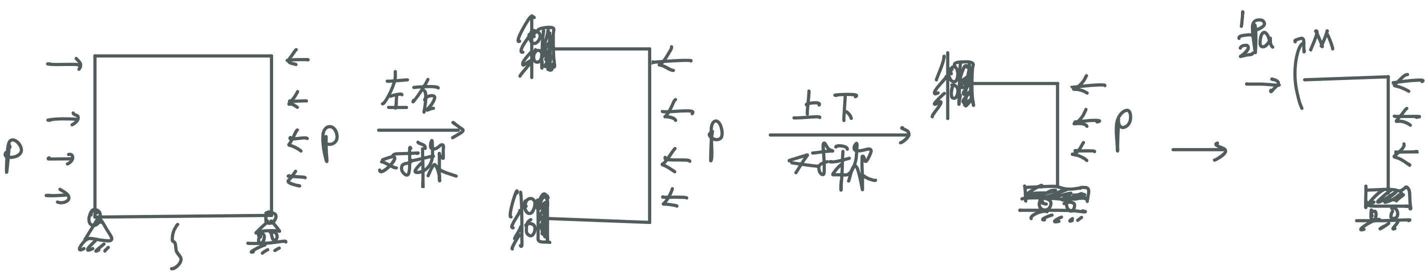


上下对称 → $N=-\frac{1}{2}a$

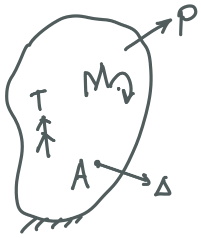
右左对称 → $Q=0$

最后由对称性可知 $\theta = \frac{\partial \phi}{\partial M} = 0 \rightarrow M = \frac{1}{24}Pa^2$

一般对称可直接采用活动固支:



§8.4. 位移积分

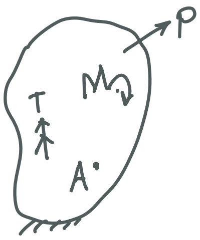


考虑一线弹性结构受一系列广义力 Q_i，若感兴趣在A点的广义位移

我们之前的做法是引入虚拟载荷 Q_A，求解 Q_i, Q_A 共同存在时

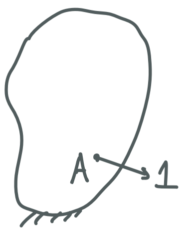
的余能 ϕ，然后 $\Delta = \left.\frac{\partial \phi}{\partial Q_A}\right|_{Q_A=0}$.

现在，我们可以利用"互易"性质来解线弹性结构在A点处的广义位移

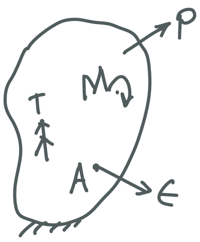


考虑一结构，其受外力 M, P, T 等一系列载荷，产生内力为 $N, M_x(x), M_z(x)$

对于线弹性结构，$\phi = \sum \frac{N^2 l}{2EA} + \sum \int_0^l \frac{M_x^2}{2GI_p} dx + \sum \int_0^l \frac{M_z^2}{2EI_z} dx$



考虑同一结构，其只在A处受一单位广义力（其方向与感兴趣的广义位移一致）。在该单位力作用下，结构的内力为 N^A, $M_x^A(x)$, $M_z^A(x)$



考虑同一结构，其不仅受外力 M, P, T 等，还在A点受一无限小广义力 ϵ（其方向与单位广义力一致）。此时结构的内力为 $N+\epsilon N^A$，$M_x+\epsilon M_x^A$，$M_z+\epsilon M_z^A$，余能为

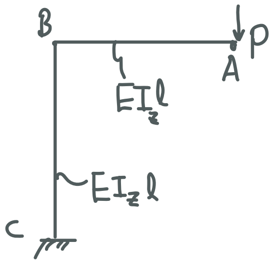
$$\phi = \sum \frac{(N+\epsilon N^A)^2 l}{2EA} + \sum \int_0^l \frac{(M_x+\epsilon M_x^A)^2}{2GI_p} dx + \sum \int_0^l \frac{(M_z+\epsilon M_z^A)^2}{2EI_z} dx$$

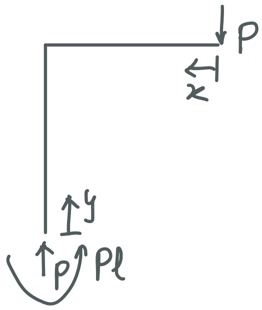
根据卡氏定理

$$\boxed{\Delta_A = \left.\frac{\partial \phi}{\partial \epsilon}\right|_{\epsilon \to 0} = \sum \frac{N \cdot N^A l}{2EA} + \sum \int_0^l \frac{M_x M_x^A}{GI_p} dx + \sum \int_0^l \frac{M_z M_z^A}{EI_z} dx}$$

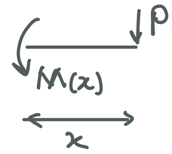
位移积分/莫尔积分

例：求A点处的①垂直位移δ_v，②水平位移δ_H，③转角θ.



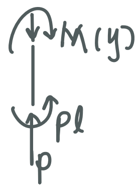


AB段：



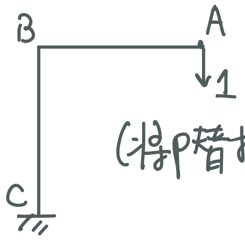
$M(x) = Px$

BC段：



$M(y) = Pl$

①



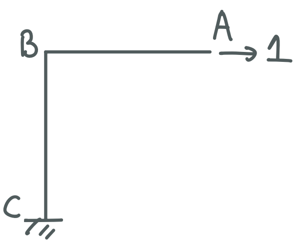
(将P替换为1)

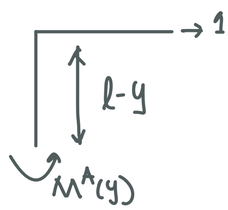
AB段：$M^A(x) = x$

BC段：$M^A(y) = l$

$$\delta_v = \int_0^l \frac{Px \cdot x}{EI_z}dx + \int_0^l \frac{Pl \cdot l}{EI_z}dy = \frac{4}{3}\frac{Pl^3}{EI_z}$$

②





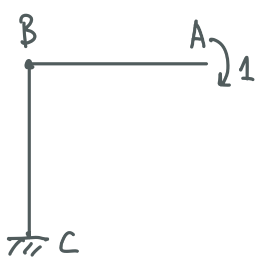
注意方向

AB段：$M^A(x) = 0$

BC段：$M^A(y) = l - y$

$$\delta_H = \int_0^l \frac{Pl(l-y)}{EI_z}dy = \frac{1}{2}\frac{Pl^3}{EI_z}$$

③

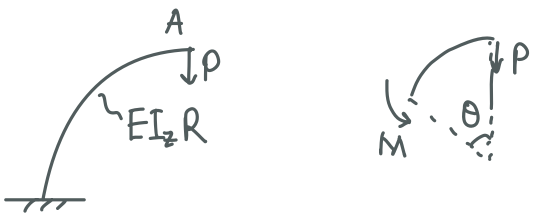


AB段：$M^A(x) = 1$

BC段：$M^A(y) = 1$

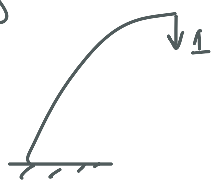
$$\theta = \int_0^l \frac{Px \cdot 1}{EI_z}dx + \int_0^l \frac{Pl \cdot 1}{EI_z}dx = \frac{3}{2}\frac{Pl^2}{EI_z}$$

例：可推广到线性曲梁，P335 求 ① δ_v ② δ_H at A



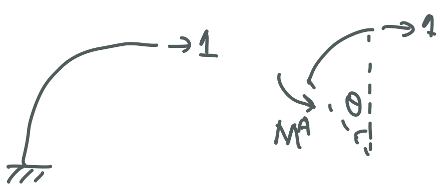
$\rightarrow M(\theta) = PR\sin\theta$

①



$M^A(\theta) = R\sin\theta$，$\delta_v = \int_0^{\frac{\pi}{2}} \frac{PR^2\sin^2\theta}{EI_z} R\,d\theta = \frac{\pi}{4}\frac{PR^3}{EI_z}$

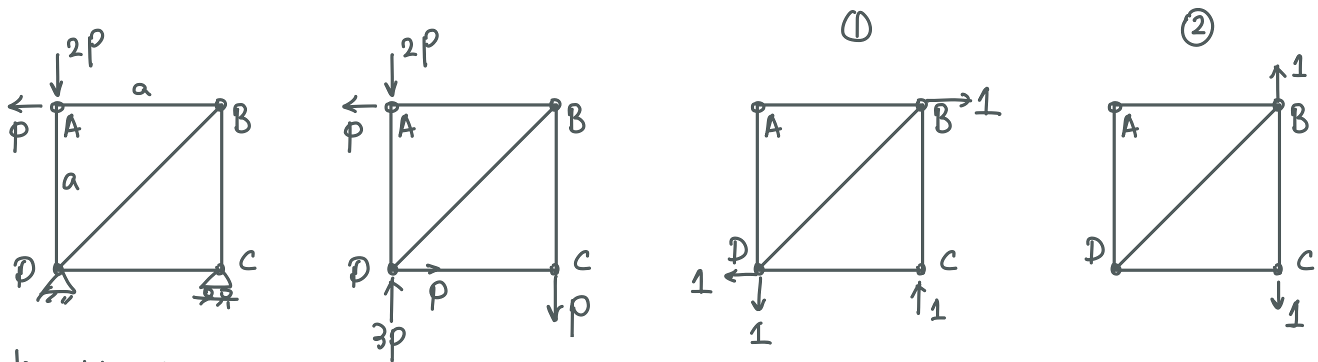
②



$M^A(\theta) = R(1-\cos\theta)$

$\delta_H = \int_0^{\frac{\pi}{2}} \frac{PR^2\sin\theta(1-\cos\theta)}{EI_z} R\,d\theta = \frac{PR^3}{2EI_z}$

例：桁架



求B处 ① δ_v ② δ_H

杆	长度 l	内力 N	内力 $N^{①}$	内力 $N^{②}$
AB	a	P	0	0
BC	a	P	-1	1
CD	a	0	0	0
DA	a	$-2P$	0	
BD	$\sqrt{2}a$	$-\sqrt{2}P$	$\sqrt{2}$	

$\delta_H = \sum \frac{NN^{①}l}{EA} = \frac{Pa}{EA}(-1-2\sqrt{2})$

$\delta_v = \sum \frac{NN^{②}l}{EA} = \frac{Pa}{EA}(1)$

§8.5. 最小势能原理

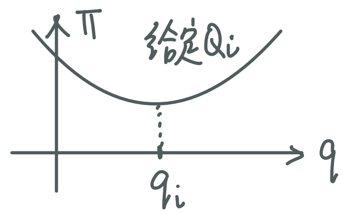
我们在欧拉失稳章节中讨论了系统总势能，简单说明了系统的解答是势能的最小值。现在我们进一步将这一原理推广到（非线性）弹性结构。

$$\Pi = U(\Delta_i, \theta_j, \varphi_k, w_l \cdots) - \Sigma(P_i\Delta_i - M_j\theta_j - T_k\varphi_k - q_l w_l \cdots)$$

$$= U(q_i) - \Sigma Q_i q_i$$

在 q_i 附近扰动

$$\Pi + \delta\Pi = U(q_i) + \Sigma \frac{\partial U}{\partial q_i}\delta q_i - \Sigma Q_i(q_i + \delta q_i)$$



$$\rightarrow \delta\Pi = \Sigma \underbrace{\left(\frac{\partial U}{\partial q_i} - Q_i\right)}_{=0\ (\text{Lagrange 原理})}\delta q_i \qquad \therefore \frac{\delta\Pi}{\delta q_i} = 0$$

在 Q_i 的作用下，满足 $\frac{\partial U}{\partial q_i} = Q_i$ 的解答（真实解答）使得 Π 取极小值。为什么极小？

考虑 $U = \frac{1}{2}EAl\left(\frac{\Delta}{l}\right)^2$，$\frac{\partial^2 U}{\partial q_i \partial q_j} \sim k > 0$.

结构刚度

最小势能原理的一个重要应用是对结构（特别是不能精确求解的结构）做近似分析。首先选择满足几何约束条件的位移场，然后确定待定参数使得总势能最小。例如，设梁的挠度为

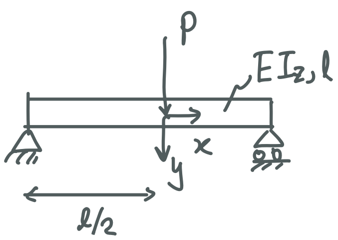
$$v = A_1 w_1(x) + A_2 w_2(x) + \cdots$$

满足边条，可以为 $x, x^2, \cdots$ 或 $\sin\frac{x}{l}, \sin\frac{2x}{l}, \cdots$ 等任意形状函数

$$\Pi = \int_0^l \frac{1}{2}EI_z (v'')^2 dx - \int_0^l p(x)\, v(x)\, dx$$

通过 $\frac{\partial\Pi}{\partial A_1}, \frac{\partial\Pi}{\partial A_2}, \cdots$ 来确定 $A_1, A_2, \cdots\cdots$（里兹 Ritz 法）.

例：



取 $v(x) = A\cos\frac{\pi x}{l}$

$$U = \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{1}{2}EI_z (v'')^2 dx$$

$$= \frac{1}{2}A^2\frac{\pi^4}{l^4}EI_z\int_{-\frac{l}{2}}^{\frac{l}{2}} \cos^2\frac{\pi x}{l}\, dx$$

$$= \frac{\pi^4}{4}EI_z\frac{A^2}{l^3}$$

真实解：$\hat{\delta}_{max} = v(0) = \frac{Pl^3}{48EI_z}$

（教材 P227 或讲义 P65）

$$\Pi = \frac{\pi^4}{4}EI_z\frac{A^2}{l^3} - P\cdot \underset{v(0)}{A} \rightarrow \frac{\partial\Pi}{\partial A} = \frac{\pi^4}{2}EI_z\frac{A}{l^3} - P = 0 \rightarrow \delta_{max} = A = \frac{2}{\pi^4}\frac{Pl^3}{EI_z}$$

$$\text{Error} = \frac{\delta_{max} - \hat{\delta}_{max}}{\hat{\delta}_{max}} = -1.45\%$$

如何提高精度？

• 试用 $v = A\cos\frac{\pi x}{l} + B\cos\frac{3\pi x}{l}$.

$$\Pi = \frac{1}{2}EI_z\int_{-l/2}^{l/2} (v'')^2 dx - P\, v(x=0)$$

$$= \frac{1}{2}\frac{\pi^4}{l^4}EI_z\int_{-l/2}^{l/2}\Big(A^2\underbrace{\cos^2\frac{\pi x}{l}}_{1/2} + \underbrace{18AB\cos\frac{\pi x}{l}\cos\frac{3\pi x}{l}}_{0} + 81B^2\underbrace{\cos^2\frac{3\pi x}{l}}_{1/2}\Big)dx - P(A+B)$$

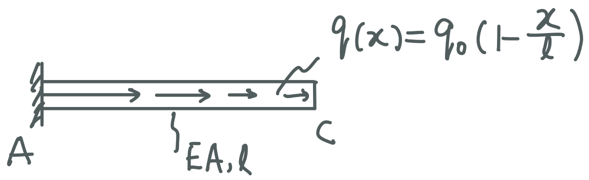
$$= \frac{\pi^4}{4}\frac{EI_z}{l^3}(A^2 + 81B^2) - P(A+B).$$

$$\left.\begin{aligned}\frac{\partial \Pi}{\partial A}=0 &\rightarrow A=\frac{2}{\pi^4}\frac{Pl^3}{EI_z}\\ \frac{\partial \Pi}{\partial B}=0 &\rightarrow B=\frac{2}{81\pi^4}\frac{Pl^3}{EI_z}\end{aligned}\right\}\rightarrow \delta_{max}=A+B=\frac{164}{81\pi^4}\frac{Pl^3}{EI_z},\quad \frac{\delta_{max}-\tilde{\delta}_{max}}{\tilde{\delta}_{max}}=-0.23\%$$

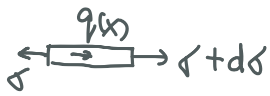
可以猜测 $v=A\cos\frac{\pi x}{l}+B\cos\frac{3\pi x}{l}+C\cos\frac{5\pi x}{l}+D\cos\frac{7\pi x}{l}$

$$\delta_{max}=\frac{2}{\pi^4}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}\right),\quad \frac{\delta_{max}-\tilde{\delta}_{max}}{\tilde{\delta}_{max}}=-0.031\%$$

例：有限元方法



真实解答：

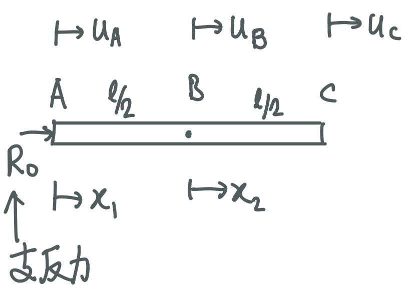


$$\frac{d\sigma}{dx}=-\frac{q_0}{A}\left(1-\frac{x}{l}\right)$$

$$\rightarrow \sigma=-\frac{q_0x}{A}+\frac{q_0x^2}{2Al}+\frac{q_0l}{2A}$$ ← 使 $\sigma(l)=0$ ✓

$$\varepsilon=-\frac{q_0x}{EA}+\frac{q_0x^2}{2EAl}+\frac{q_0l}{2EA}$$

$$u=-\frac{q_0x^2}{2EA}+\frac{q_0x^3}{6EAl}+\frac{q_0lx}{2EA}+\cancel{u_0}^{\,0}\rightarrow\begin{cases}u(l/2)=\underbrace{\left(-\frac{1}{8}+\frac{1}{48}+\frac{1}{4}\right)}_{\frac{7}{48}}\frac{q_0l^2}{EA}\\ u(l)=\frac{q_0l^2}{6EA}\end{cases}$$



单元① $u=\left(1-\frac{2x_1}{l}\right)u_A+\frac{2x_1}{l}u_B$

$$\varepsilon_{①}=\frac{2(u_B-u_A)}{l}$$

单元② $u=\left(1-\frac{2x_2}{l}\right)u_B+\frac{2x_2}{l}u_C$

$$\varepsilon_{②}=\frac{2(u_C-u_B)}{l}$$

$$\Pi = \int_0^{\frac{l}{2}} \frac{1}{2} EA\, \varepsilon_{①}^2\, dx + \int_0^{\frac{l}{2}} \frac{1}{2} EA\, \varepsilon_{②}^2\, dx - \int_0^{l} q\, u\, dx - R_0 u_A$$

$$\delta\Pi = \int_0^{l/2} EA\, \varepsilon_{①}\, \delta\varepsilon_{①}\, dx + \int_0^{l/2} EA\, \varepsilon_{②}\, \delta\varepsilon_{②}\, dx - \int_0^{l/2} q\, \delta u\, dx_1 - \int_0^{\frac{l}{2}} q\, \delta u\, dx_2 - R_0 \delta u_A$$

$$= EA \int_0^{\frac{l}{2}} \frac{4}{l^2}(u_B - u_A)(\delta u_B - \delta u_A)\, dx_1 + EA \int_0^{l/2} \frac{4}{l^2}(u_C - u_B)(\delta u_C - \delta u_B)\, dx_2$$

$$- \int_0^{\frac{l}{2}} q_0 \Big(1 - \underbrace{\frac{x_1}{l}}_{x = x_1}\Big) \Big[\Big(1 - \frac{2x_1}{l}\Big) \delta u_A + \frac{2x_1}{l} \delta u_B \Big] dx_1$$

$$- \int_0^{\frac{l}{2}} q_0 \Big(1 - \underbrace{\frac{x_2 + l/2}{l}}_{x = x_2 + \frac{l}{2}}\Big) \Big[\Big(1 - \frac{2x_2}{l}\Big) \delta u_B + \frac{2x_2}{l} \delta u_C \Big] dx_2 - R_0 \delta u_A$$

边界约束：$u_A \equiv 0$，$\delta u_A = 0$，可简化上述结果为：

$$\delta\Pi = \delta u_B \left(\frac{2EA}{l} u_B - \frac{2EA}{l}(u_C - u_B) - \frac{1}{6} q_0 l - \frac{1}{12} q_0 l \right) + \delta u_C \left(\frac{2EA}{l}(u_C - u_B) - \frac{1}{24} q_0 l \right)$$

$$\frac{\delta\Pi}{\delta u_B} = 0 \ \ \& \ \ \frac{\delta\Pi}{\delta u_C} = 0 \ \rightarrow \ u_B = \frac{7 q_0 l^2}{48 EA}, \quad u_C = \frac{q_0 l^2}{6EA}$$ （可通过 δu_A 项求 R_0）

节点上位移与真实解相同!!!

(仅限于任何非奇异1维杆梁问题)

注意该近似解在应变、位移分布上并不准确.