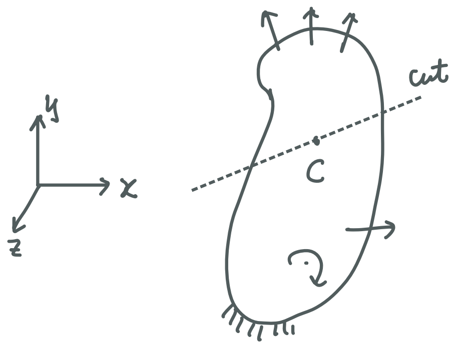
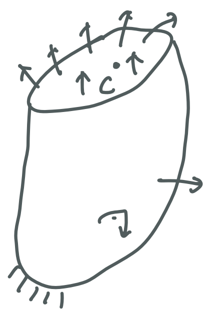
第七章 应力应变分析&强度"理论"

§7.1. 应力状态

在这一章，我们进一步关注构件内部内力的分布，也就是在某一点处单位面积的内力。

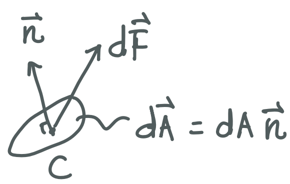


比如，我们想知道C点"附近"的内力。为此，我们需要做过P点的切面(cut)。



做切面后，需用面与面之间的作用力替代被切的部分产生的作用。该作用力的方向和大小均是位置的函数。

若感兴趣C点位置的内力分布，则考查C点附近的微元。



$\vec n$: 面元的外法向单元向量

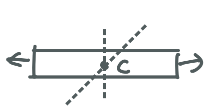
dA: 面元的面积

$d\vec F$: 作用于面元的合力

量级 ~ [力]/[面积]

→ 定义作用力矢量(Traction): $\vec P_n = \lim\limits_{dA\to 0} \dfrac{d\vec F}{dA} = X_n\vec i + Y_n\vec j + Z_n\vec k$

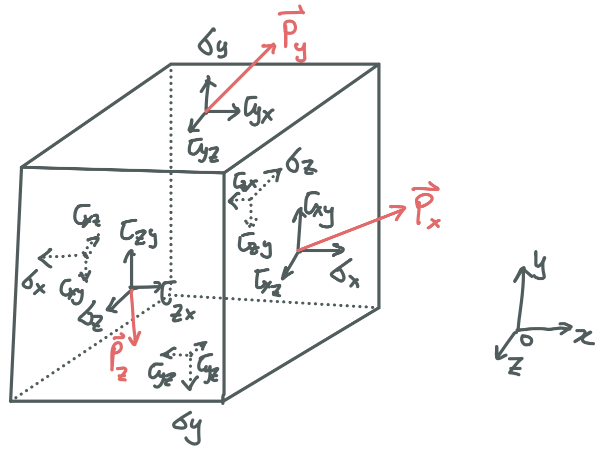
下标n代表由外法向为n的切面所"暴露"出来的作用力。显然n不同，$\vec P_n$不同



(例如)

全面了解C点处的内力是否要进行无数的切面，得到所有n方向对应的$\vec P_n$?

答案：不是，只需做3个正交(或不相关)的切面即可。



设法向为 $\vec{i}$ 的切面上： $\vec{P}_x = \sigma_x \vec{i} + \tau_{xy} \vec{j} + \tau_{xz} \vec{k}$

…… $\vec{j}$ …… ： $\vec{P}_y = \tau_{yx} \vec{i} + \sigma_y \vec{j} + \tau_{yz} \vec{k}$

…… $\vec{k}$ …… ： $\vec{P}_z = \tau_{zx} \vec{i} + \tau_{zy} \vec{j} + \sigma_z \vec{k}$

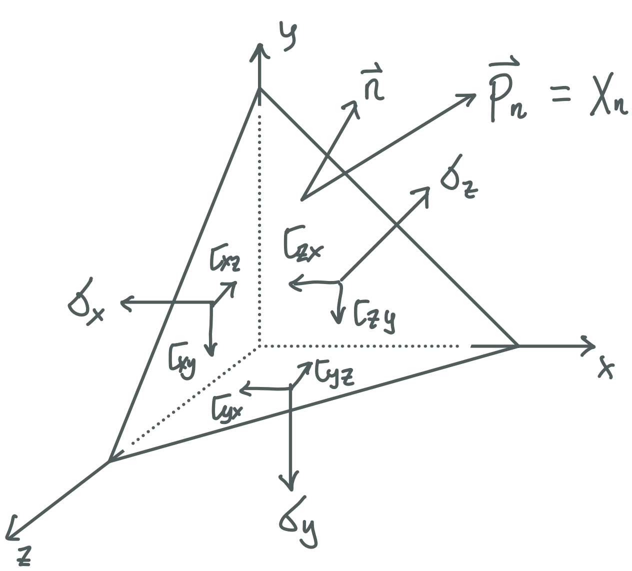
显然，在 $-\vec{i}$ 方向切面上 $\vec{P}_{-x} = -\vec{P}_x$，同样 $\vec{P}_{-y} = -\vec{P}_y$，$\vec{P}_{-z} = -\vec{P}_z$ (大小相等，方向相反)。

• τ_{ij} ← 力矢量的分量 $\left\{ \begin{array}{l} i\text{为正，}j\text{指向正} \\ i\text{为负，}j\text{指向负} \end{array} \right.$ → 力平衡自动满足 ($\Sigma \vec{F} = 0$)

作用面法向指向

• 力矩平衡 $\left\{ \begin{array}{l} \Sigma M_x = 0 \rightarrow \tau_{yz} = \tau_{zy} \\ \Sigma M_y = 0 \rightarrow \tau_{xz} = \tau_{zx} \\ \Sigma M_z = 0 \rightarrow \tau_{xy} = \tau_{yx} \end{array} \right.$ → $\tau_{ij} = \tau_{ji}$ (切应力互等)

假设已知 $\vec{P}_x, \vec{P}_y, \vec{P}_z$，考虑过该点的任意切面，其法向为 $\vec{n} = n_x \vec{i} + n_y \vec{j} + n_z \vec{k}$，应力矢量为 $\vec{P}_n$。将该切面与法向为 $-\vec{i}, -\vec{j}, -\vec{k}$ 的切面组成“四面体”(体积→0)。



$\vec{P}_n = X_n \vec{i} + Y_n \vec{j} + Z_n \vec{k}$

ΣF_x: $\sigma_x \cdot A_x + \tau_{yx} A_y + \tau_{zx} \cdot A_z = X_n \cdot A$

几何: $\dfrac{A_x}{A} = n_x$, $\dfrac{A_y}{A} = n_y$, $\dfrac{A_z}{A} = n_z$

$\rightarrow X_n = \sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z$

$\rightarrow Y_n, Z_n$ (同样)

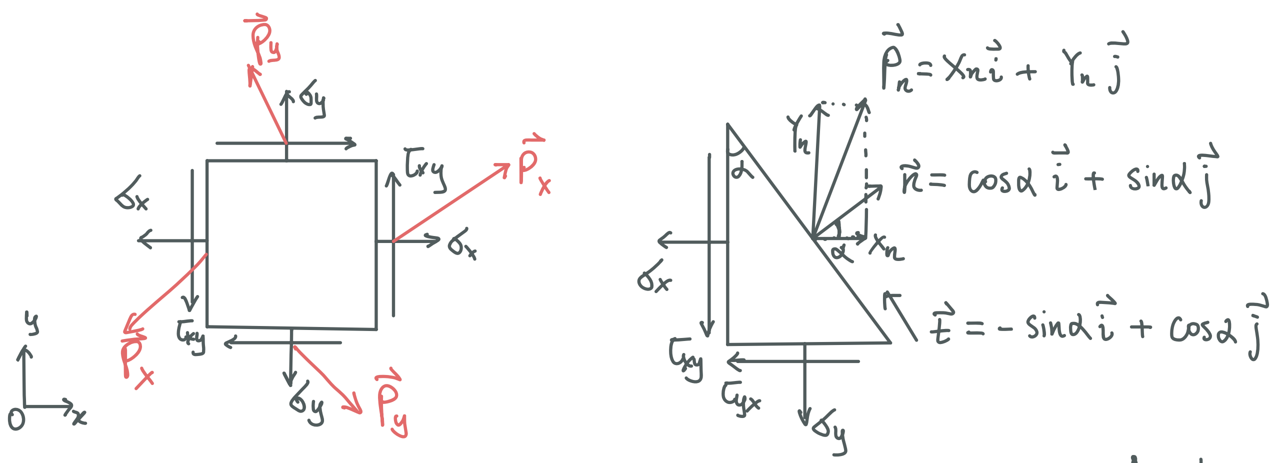
$$\begin{bmatrix} X_n \\ Y_n \\ Z_n \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}}_{} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \rightarrow \vec{p} = \Sigma \vec{n}$$

Σ: 二阶应力张量的矩阵表示

若我们知道Σ中9个分量，就可以得到任意$\vec{n}$切面上的内力状态(应力矢量) i.e., Σ可以表征一点的应力状态.

§ 7.2. 平面应力状态

为了更清楚的了解应力张量的性质，我们考虑简单的平面应力状态，也就是 $\tau_{xz} = \tau_{yz} = \sigma_z = 0$ 时应力张量.



$$\begin{bmatrix} X_n \\ Y_n \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{yx} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

或

$$\Sigma F_x = X_n A - \sigma_x \overset{A\cos\alpha}{A_x} - \tau_{yx} \overset{A\sin\alpha}{A_y} = 0$$

$$\Sigma F_y = Y_n A - \tau_{xy} A_x - \sigma_y A_y = 0$$

也可以将$\vec{P}_n$在$\vec{n}$和$\vec{t}$方向投影(分解), i.e., $\vec{P}_n = \sigma_n \vec{n} + \tau_{nt} \vec{t}$

$$\sigma_n = X_n \cos\alpha + Y_n \sin\alpha$$

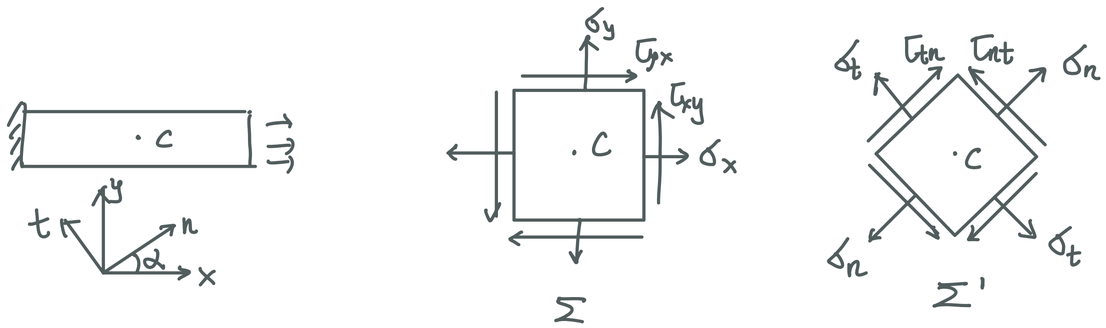
$$\tau_{nt} = -X_n \sin\alpha + Y_n \cos\alpha$$

或

$$\begin{bmatrix} \sigma_n \\ \sigma_{nt} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{yx} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

一旦确定oxy坐标系下的某一点处的应力状态Σ(或 $\vec{P}_x$，$\vec{P}_y$)，便可确定 $\underset{\substack{\uparrow\\ \text{任意方向}}}{\vec{P}_n}$.

在另一视角或坐标系下，如何表征该点的应力状态Σ'？Σ'与Σ如何关联？



可借用 $\begin{bmatrix}\sigma_n\\ \tau_{nt}\end{bmatrix}$ 与Σ的关系来确定 $\begin{bmatrix}\sigma_t\\ \tau_{tn}\end{bmatrix}$ 与Σ的关系！但注意 $\alpha\to\alpha+\frac{\pi}{2}$，$\sigma_n\to\sigma_t$，$\underbrace{\tau_{nt}\to-\tau_{tn}}_{\text{方向定义相反}}$

$$\to\begin{bmatrix}\sigma_t\\ -\sigma_{tn}\end{bmatrix}=\begin{bmatrix}-\sin\alpha & \cos\alpha\\ -\cos\alpha & -\sin\alpha\end{bmatrix}\begin{bmatrix}\sigma_x & \tau_{yx}\\ \tau_{xy} & \sigma_y\end{bmatrix}\begin{bmatrix}-\sin\alpha\\ \cos\alpha\end{bmatrix}$$

$$\begin{bmatrix}a\\ -b\end{bmatrix}=\begin{bmatrix}-\sin\alpha & \cos\alpha\\ -\cos\alpha & -\sin\alpha\end{bmatrix}\begin{bmatrix}c\\ d\end{bmatrix}$$

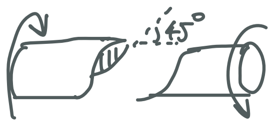
$$\text{或}\begin{bmatrix}\sigma_{tn}\\ \sigma_t\end{bmatrix}=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\sigma_x & \tau_{yx}\\ \tau_{xy} & \sigma_y\end{bmatrix}\begin{bmatrix}-\sin\alpha\\ \cos\alpha\end{bmatrix}$$

$$\to\begin{bmatrix}\sigma_n & \tau_{tn}\\ \tau_{nt} & \sigma_t\end{bmatrix}=\underbrace{\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}}_{A}\begin{bmatrix}\sigma_x & \tau_{yx}\\ \tau_{xy} & \sigma_y\end{bmatrix}\underbrace{\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}}_{A^T\ (A\text{的转置})}\quad\text{或}\quad\boxed{\Sigma'=A\Sigma A^T}$$

$$\boxed{\begin{aligned}\sigma_n&=\frac{1}{2}(\sigma_x+\sigma_y)+\frac{1}{2}(\sigma_x-\sigma_y)\cos 2\alpha+\tau_{xy}\sin 2\alpha\\ \sigma_t&=\frac{1}{2}(\sigma_x+\sigma_y)-\frac{1}{2}(\sigma_x-\sigma_y)\cos 2\alpha-\tau_{xy}\sin 2\alpha\\ \tau_{nt}&=\tau_{tn}=-\frac{1}{2}(\sigma_x-\sigma_y)\sin 2\alpha+\tau_{xy}\cos 2\alpha\end{aligned}}$$

平面应力状态
张量变换公式

注意到 $\underbrace{\sigma_n+\sigma_t}_{\text{不变量①}}=\sigma_x+\sigma_y$ → 不同坐标系下的应力状态中，正应力之和为常数



铸铁(骨头)

在扭转问题中我们讨论过，当 $\alpha=0°$ 时只有切应力

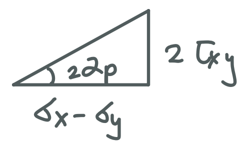
当 $\alpha\neq 0°$ 时，切面存在正应力（$\alpha=45°$ 时，正应力最大）

Σ' 中正应力分量显然依赖于 α. 现在讨论正应力的最大值和最小值，i.e., 主应力.

发生主应力的切面为主平面，其外法向与 x 轴夹角为 α_p.

$$\left.\frac{d\sigma_n}{d\alpha}\right|_{\alpha=\alpha_p} = -(\sigma_x-\sigma_y)\sin 2\alpha + 2\tau_{xy}\cos 2\alpha = 0 \rightarrow \tan 2\alpha_p = \frac{2\tau_{xy}}{\sigma_x-\sigma_y}$$

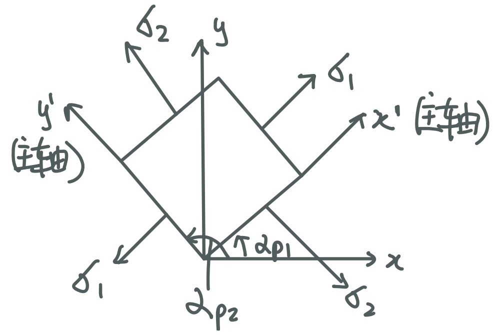
$$\rightarrow 2\alpha_p = \underbrace{\arctan\frac{2\tau_{xy}}{\sigma_x-\sigma_y}}_{2\alpha_{p1}} \text{ 或 } \underbrace{\arctan\frac{2\tau_{xy}}{\sigma_x-\sigma_y}+\pi}_{2\alpha_{p2}}$$



$$\rightarrow \cos 2\alpha_p = \pm\frac{\sigma_x-\sigma_y}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}, \quad \sin 2\alpha_p = \pm\frac{2\tau_{xy}}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}$$

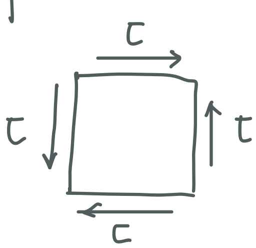
$$\rightarrow \sigma_1 = \sigma_n\Big|_{\alpha_{p1}} = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2} \quad \text{最大值}$$

$$\sigma_2 = \sigma_n\Big|_{\alpha_{p2}} = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2} \quad \text{最小值}$$



$$\tau_{nt}\Big|_{\alpha_{p1}} = \tau_{nt}\Big|_{\alpha_{p2}} = 0 \quad \text{(主平面上切应力为0)}$$

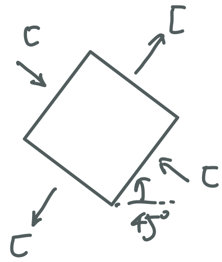
例:



$$2\alpha_p = \arctan\frac{2\tau}{0} = \frac{\pi}{2}$$

$$\alpha_p = \frac{\pi}{4}$$

$$\sigma_{1,2} = 0 \pm \sqrt{\tau^2} = \pm\tau$$



进一步考查 切应力最大/小值及对应的切面方向 α_s（与 x 轴夹角）

$$\left.\frac{d\tau_{nt}}{d\alpha}\right|_{\alpha_s} = -(\sigma_x-\sigma_y)\cos 2\alpha_s - 2\tau_{xy}\sin 2\alpha_s = 0 \rightarrow \underbrace{\cot 2\alpha_s = -\frac{2\tau_{xy}}{\sigma_x-\sigma_y} = -\tan 2\alpha_p}_{\alpha_s \text{ 与 } \alpha_p \text{ 相差 } \pm 45^\circ}$$

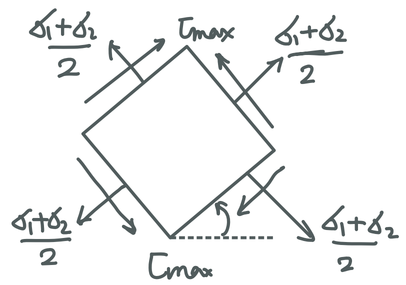
当 $2\alpha = 2\alpha_s$ 时，得到

$$\tau = \tau_{max} = \pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2} = \pm\frac{\sigma_1-\sigma_2}{2}$$

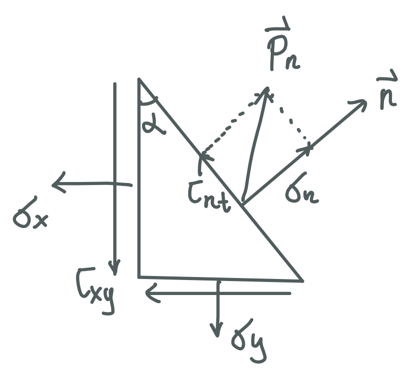
$$\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2} = \sqrt{\underbrace{\left(\frac{\sigma_x+\sigma_y}{2}\right)^2}_{\text{不变量①}} - \underbrace{(\sigma_x\sigma_y-\tau_{xy}^2)}_{\text{不变量②}}}$$

$$\sigma = \frac{\sigma_x+\sigma_y}{2} = \underbrace{\frac{\sigma_1+\sigma_2}{2}}_{\text{通常}\neq 0}$$

且 τ_{max} 不依赖坐标系



§7.3. 应力圆



给定应力状态Σ，考查在外法向为 $\vec{n}$ 截面上的 $\vec{P}_n$

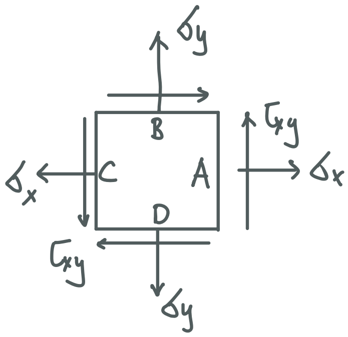
现在定义 $\sigma = \sigma_n$，$\tau = -\tau_{nt}$（与教材保持统一）

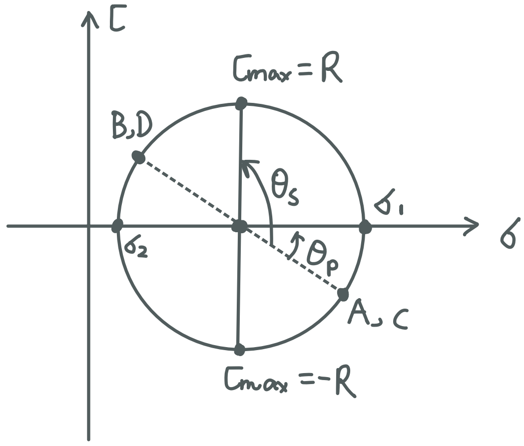
$$\rightarrow \begin{aligned} \sigma &= \tfrac{1}{2}(\sigma_x+\sigma_y) + \tfrac{1}{2}(\sigma_x-\sigma_y)\cos 2\alpha + \tau_{xy}\sin 2\alpha \\ \tau &= +\tfrac{1}{2}(\sigma_x-\sigma_y)\sin 2\alpha - \tau_{xy}\cos 2\alpha \end{aligned}$$

$$\left(\sigma-\frac{\sigma_x+\sigma_y}{2}\right)^2 = \left(\frac{\sigma_x-\sigma_y}{2}\right)^2\cos^2 2\alpha + \tau_{xy}^2\sin^2 2\alpha + (\sigma_x-\sigma_y)\tau_{xy}\sin 2\alpha\cos 2\alpha$$

$$\tau^2 = \left(\frac{\sigma_x-\sigma_y}{2}\right)^2\sin^2 2\alpha + \tau_{xy}^2\cos^2 2\alpha - (\sigma_x-\sigma_y)\tau_{xy}\sin 2\alpha\cos 2\alpha$$

$$\longrightarrow \underbrace{\left(\sigma-\frac{\sigma_x+\sigma_y}{2}\right)^2}_{(x-x_c)^2}+\underbrace{\tau^2}_{y^2}=\underbrace{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}_{R^2}$$





A: σ_x, $-\tau_{xy}$

B: σ_y, τ_{xy}

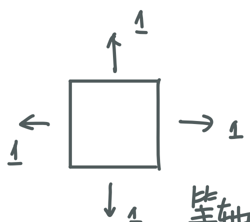
C 与 A 相同

D 与 B 相同

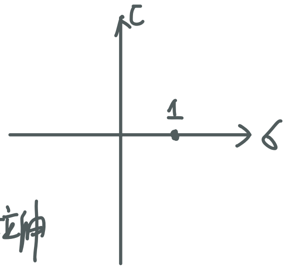
给定任意应力状态，以 $\frac{\sigma_x+\sigma_y}{2}$ 为圆心，$\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$ 为半径做应力圆（莫尔圆）

- 不同方切面应力分量 σ_n, τ_{nt} 为圆上的点. ($\sigma=\sigma_n$, $\tau=-\tau_{nt}$)
- $\sigma_{1,2}=x_c\pm R=\frac{\sigma_x+\sigma_y}{2}\pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$
- 方相差 $90°$ 的切面对应在莫尔圆上的点相差 $180°$（AB 或 CD 过圆心）
- 在莫尔圆逆时针转动 θ，对应切面法向逆时针转动 $\frac{1}{2}\theta$.

例： $\sin\theta_p=\dfrac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}}=\sin 2\alpha_p$; $\alpha_s=\alpha_p+\frac{\pi}{4}\Leftrightarrow\theta_s=\theta_p+\frac{\pi}{2}$

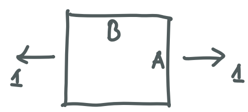


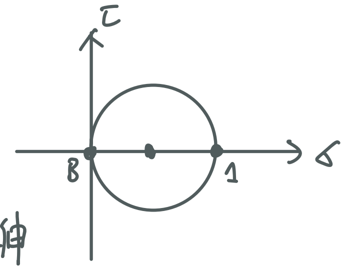
等轴拉伸



$R=0$

$x_c=1$

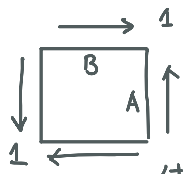




单轴拉伸

$R=\frac{1}{2}$

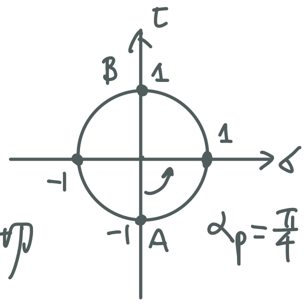
$x_c=\frac{1}{2}$



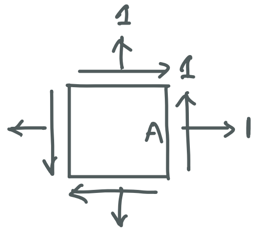
纯剪切

$x_c = 0$

$R = 1$

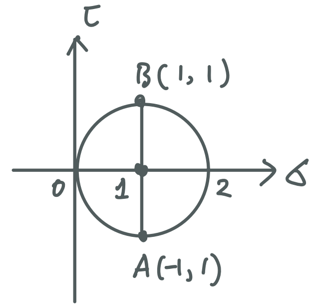


$\alpha_p = \frac{\pi}{4}$

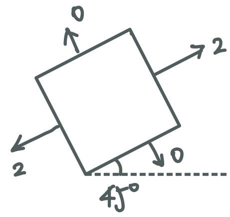


$x_c = 1$

$R = 1$



⇓ 很容易得出



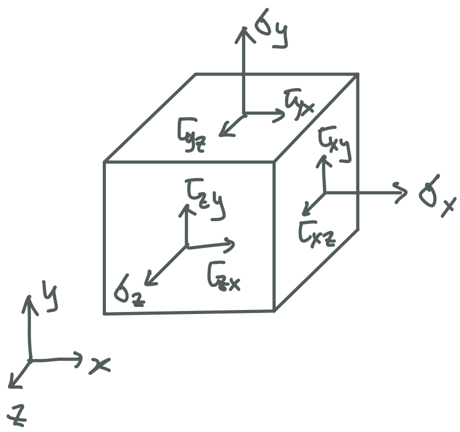
$\tau_{max} = 1$, $\alpha_s = 0, 90^\circ$

$\sigma_1 = 2$, $\alpha_{p_1} = 45^\circ$

$\sigma_2 = 0$, $\alpha_{p_2} = -45^\circ$ 或 135°

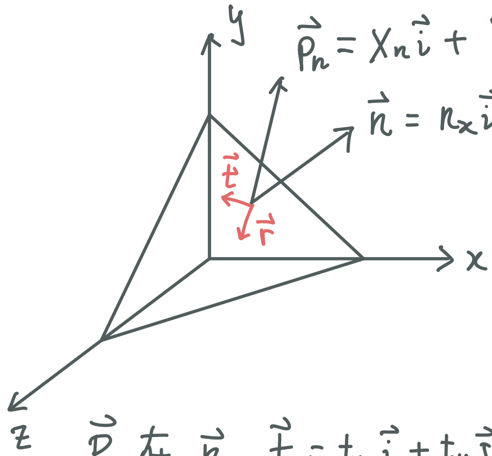
§ 7.4 空间应力状态

现在回到更一般形式的三维应力状态，在上一节讨论的张量性质可以相应的推广。



$$\Sigma = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

共有6个独立分量.



$\vec{P}_n = X_n \vec{i} + Y_n \vec{j} + Z_n \vec{k}$

$\vec{n} = n_x \vec{i} + n_y \vec{j} + n_z \vec{k}$

已通过平衡证明 $\vec{P}_n = \Sigma \vec{n}$

$$\begin{bmatrix} X_n \\ Y_n \\ Z_n \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

$\vec{P}_n$ 在 $\vec{n}$, $\vec{t} = t_x \vec{i} + t_y \vec{j} + t_z \vec{k}$, $\vec{r} = r_x \vec{i} + r_y \vec{j} + r_z \vec{k}$ 分量？或 ontr坐标 的Σ'？

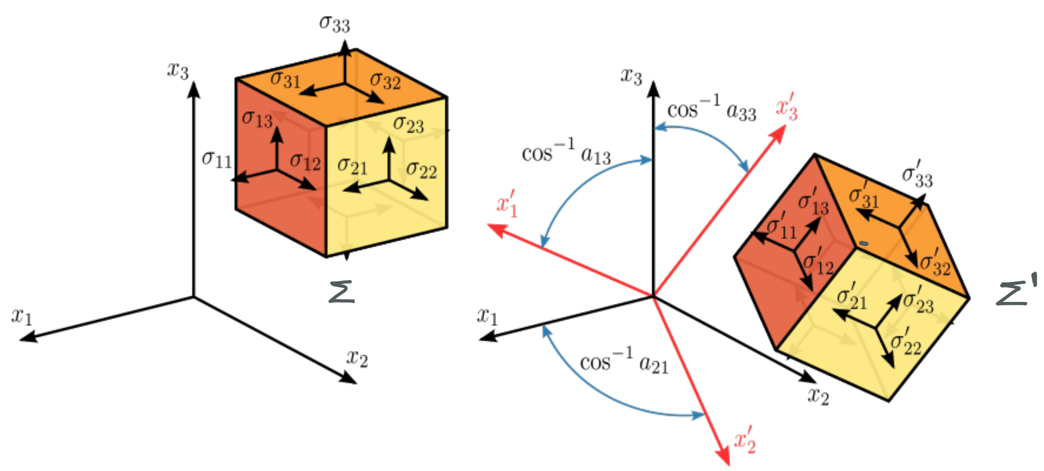
我们在$\underbrace{\text{平面}}_{2\times 2}$应力状态下得出$\Sigma' = A\Sigma A^T$，该公式可推广到$3\times3$矩阵表示的三维应力状态.

首先考查 $\sigma_n = \vec{P}_n \cdot \vec{n} = \begin{bmatrix} n_x & n_y & n_z \end{bmatrix}\begin{bmatrix} X_n \\ Y_n \\ Z_n \end{bmatrix}$

$$\tau_{nt} = \vec{P}_n \cdot \vec{t} = \begin{bmatrix} t_x & t_y & t_z \end{bmatrix}\begin{bmatrix} X_n \\ Y_n \\ Z_n \end{bmatrix}, \quad \tau_{nr} = \begin{bmatrix} r_x & r_y & r_z \end{bmatrix}\begin{bmatrix} X_n \\ Y_n \\ Z_n \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \sigma_n \\ \tau_{nt} \\ \tau_{nr} \end{bmatrix} = \begin{bmatrix} n_x & n_y & n_z \\ t_x & t_y & t_z \\ r_x & r_y & r_z \end{bmatrix}\begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

$$\rightarrow \underbrace{\begin{array}{ccc} \vec{n}\text{切面} & \vec{t}\text{切面} & \vec{r}\text{切面} \\ \left[\begin{matrix} \sigma_n \\ \tau_{nt} \\ \tau_{nr} \end{matrix}\right. & \begin{matrix} \tau_{tn} \\ \sigma_t \\ \tau_{tr} \end{matrix} & \left.\begin{matrix} \tau_{rn} \\ \tau_{rt} \\ \sigma_r \end{matrix}\right] \end{array}}_{\Sigma'} = \underbrace{\begin{bmatrix} n_x & n_y & n_z \\ t_x & t_y & t_z \\ r_x & r_y & r_z \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} \sigma_x & \tau_{yz} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix} n_x & t_x & r_x \\ n_y & t_y & r_y \\ n_z & t_z & r_z \end{bmatrix}}_{A^T}$$



我们继续考查主应力及主平面(主坐标系)，根据性质：主平面上切应力为0.

外法向$\vec{n}$切面为主平面，则 $\vec{P}_n = \sigma\vec{n} + \cancelto{0}{\tau_{nt}}\vec{t} + \cancelto{0}{\tau_{nr}}\vec{r} = \Sigma\vec{n}$

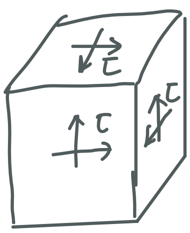
$$\rightarrow \underbrace{\begin{bmatrix} \sigma_x-\sigma & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y-\sigma & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z-\sigma \end{bmatrix}}_{C} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = 0$$ 特征值问题

$\vec{n}$不为0(平凡解)的条件为 $|C| = 0 \longrightarrow \sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$

$I_1 = \sigma_x + \sigma_y + \sigma_z$， $I_2 = \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{zx} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix}$， $I_3 = |\Sigma|$

- 特征方程可以解答出3个实根，设 $\sigma_1 \geq \sigma_2 \geq \sigma_3$ (三个主应力)
- 一个点的应力状态在不同 ontr 下形式不同，但主应力一致，因此 I_1, I_2, I_3 为不变量
- n_x, n_y, n_z 并不任意，需额外满足 $n_x^2 + n_y^2 + n_z^2 = 1$，三主应力对应三个主方向.

例



$\Sigma = \begin{bmatrix} 0 & \tau & \tau \\ \tau & 0 & \tau \\ \tau & \tau & 0 \end{bmatrix}$ 求主应力，主方向？

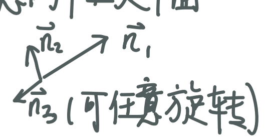
特征方程 $\sigma^3 - 0\sigma^2 + (-3\tau^2)\sigma - 2\tau^3 = \sigma^3 - 3\tau^2\sigma - 2\tau^3 = 0$

$\longrightarrow \sigma_1 = 2\tau,\ \sigma_2 = \sigma_3 = -\tau$ (重根)

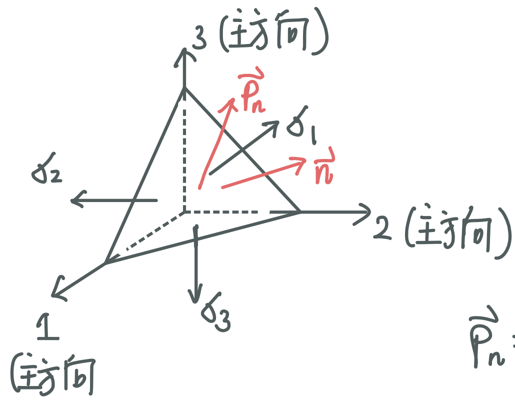
特征方向(以$\vec{n}_1$为例)

$$\left.\begin{matrix} \begin{bmatrix} -2\tau & \tau & \tau \\ \tau & -2\tau & \tau \\ \tau & 0 & -2\tau \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = 0 \\ n_x^2 + n_y^2 + n_z^2 = 1 \end{matrix}\right\} \rightarrow \vec{n}_1 = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$$

代入 $\sigma = -\tau$ 会出现只有2个线性无关方程，这代表着有无穷几个$\vec{n}_2$，$\vec{n}_3$，或与$\vec{n}_1$垂直的任意两个正交平面都是主平面



在主坐标系下，$\Sigma = \begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{bmatrix}$ 极为简单，以下采用主坐标系来描述应力状态&性质



$$\vec{n} = n_1\vec{i} + n_2\vec{j} + n_3\vec{k}$$

$$\vec{P_n} = X_n\vec{i} + Y_n\vec{j} + Z_n\vec{k}$$

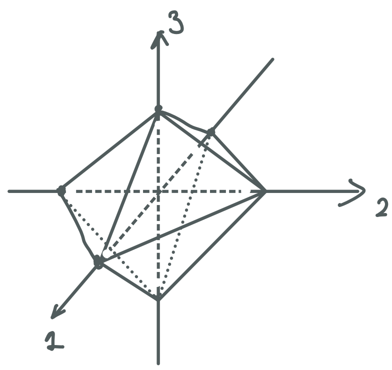
$$\vec{P_n} = \Sigma\vec{n} \rightarrow X_n = \sigma_1 n_1,\ Y_n = \sigma_2 n_2,\ Z_n = \sigma_3 n_3$$

外法向$\vec{n}$切面上的正应力 $\sigma = \vec{P_n}\cdot\vec{n} = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$ ①

外法向$\vec{n}$切面上的切应力 $\tau = \sqrt{\tau_{nt}^2 + \tau_{nr}^2} = \sqrt{|P_n|^2 - \sigma^2}$

$$= \left[\sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3 n_3^2 - (\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2)^2\right]^{1/2} \quad ②$$

· 八面体应力



$$n_1 = n_2 = n_3 = \pm\frac{1}{\sqrt{3}}$$

$$\rightarrow \sigma_8 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

$$\tau_8 = \left[\frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2\right]^{1/2}$$

$$= \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$ (也是模量?)

· 极值切应力

在 $\underbrace{n_1^2 + n_2^2 + n_3^2 = 1}_{③}$ 的约束下，求解 τ 的最大值，可采用 Lagrange multiplier.

$$f(n_1, n_2, n_3, \lambda) = \tau^2 - \lambda(n_1^2 + n_2^2 + n_3^2 - 1)$$

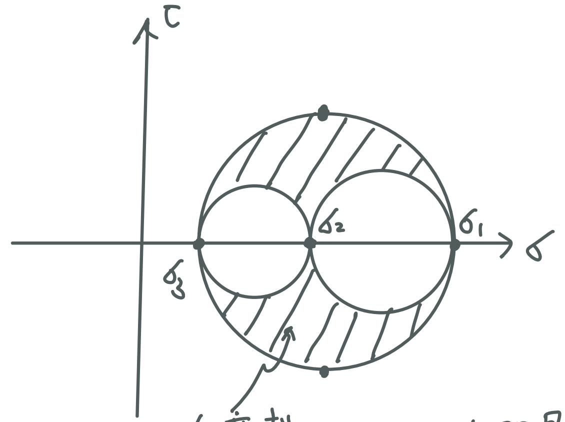
或根据方程①,②,③可得出

$$n_1^2=\frac{(\sigma-\sigma_2)(\sigma-\sigma_3)+\tau^2}{(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)}\geqslant 0 \rightarrow \left(\sigma-\frac{\sigma_2+\sigma_3}{2}\right)^2+\tau^2\geqslant\left(\frac{\sigma_2-\sigma_3}{2}\right)^2$$

同样可得 n_2^2，n_3^2 的表达式

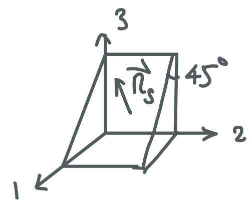
$$\left(\sigma-\frac{\sigma_1+\sigma_3}{2}\right)^2+\tau^2\leqslant\left(\frac{\sigma_1-\sigma_3}{2}\right)^2$$

$$\left(\sigma-\frac{\sigma_1+\sigma_2}{2}\right)^2+\tau^2\geqslant\left(\frac{\sigma_1-\sigma_2}{2}\right)^2$$

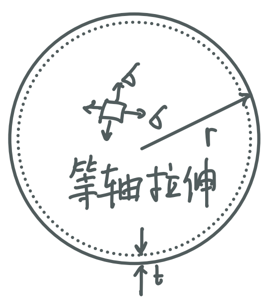


任意截面上(σ, τ)在阴影区内

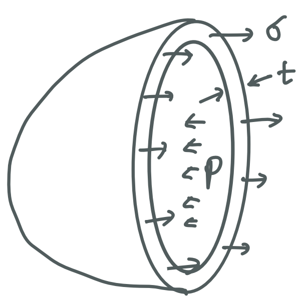
→ 最大切应力：$\boxed{\tau_{max}=\frac{\sigma_1-\sigma_3}{2}}$，对应的 $\vec{n}_s=\frac{\sqrt{2}}{2}\vec{i}+\frac{\sqrt{2}}{2}\vec{k}$



• 球状、柱状压力容器



$t\ll r$



p作用于表面法向方向，不依赖具体形状

$$2\pi r t\cdot\sigma=\pi r^2\cdot p$$

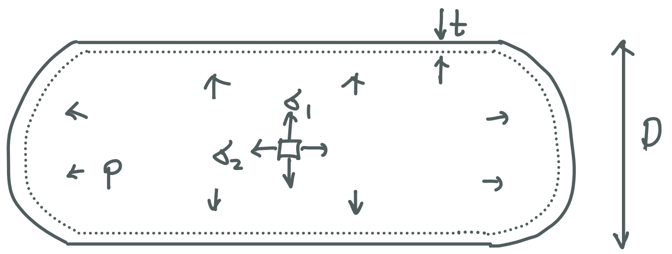
$$\rightarrow \boxed{\sigma=\frac{pr}{2t}}$$

Laplace law

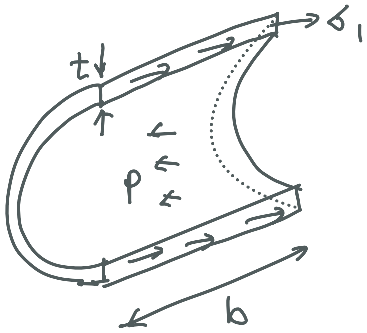
$$p=\sigma t\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

外表面 $\sigma_1=\sigma_2=\frac{pr}{2t}$，$\sigma_3=0$，$\tau_{max}=\frac{pr}{4t}$

内表面 $\sigma_1=\sigma_2=\frac{pr}{2t}$，$\sigma_3=-p$，$\tau_{max}=\frac{pr}{4t}+\frac{p}{2}\simeq\frac{pr}{4t}$



$\sigma_2 = \frac{PD}{4t}$ (与球状类似)



$2 \times bt \times \sigma_1 = P \times (bD) \rightarrow \sigma_1 = \frac{PD}{2t} = 2\sigma_2$

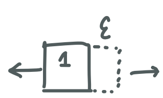
内表面：$\sigma_1 = \frac{PD}{2t}$，$\sigma_2 = \frac{PD}{4t}$，$\sigma_3 = -P$；外表面 $\sigma_1 = \frac{PD}{2t}$，$\sigma_2 = \frac{PD}{4t}$，$\sigma_3 = 0$

§7.5. 平面应变状态

和应力状态一样，在构件内一点的应力状态为二阶对称张量

$$E = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix}$$

ε - 线应变，对应

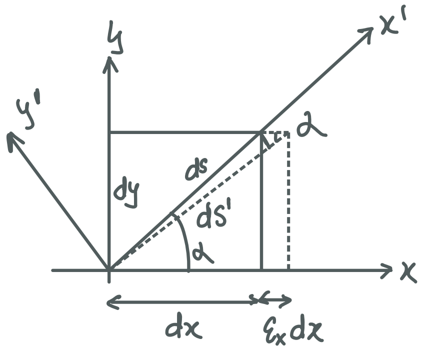


γ - 切应变，对应

我们仍然从简单的平面应变状态（$\gamma_{zx} = \gamma_{zy} = \varepsilon_z = 0$）开始讨论。我们后续会发现，由于泊松效应，平面应力 ≠ 平面应变，但 $\boldsymbol{\Sigma}$ 和 $\boldsymbol{E}$ 的性质基本相同。

考虑 oxy 坐标下平面应变状态应变分量 ε_x、ε_y，γ_{xy}，在 ox'y' 坐标系的 $\varepsilon_{x'}$，$\varepsilon_{y'}$，$\gamma_{x'y'}$？

• ε_x 对转角α处线段的影响



$$ds' = \sqrt{(1+\varepsilon_x)^2 dx^2 + dy^2} = \sqrt{dx^2 + dy^2 + 2\varepsilon_x dx^2 + O(\varepsilon_x^2)}$$

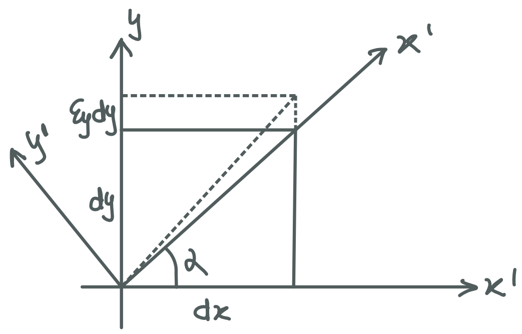
$$= ds\sqrt{1 + \frac{2\varepsilon_x dx^2}{ds^2}} = ds + \varepsilon_x \frac{dx^2}{ds}$$

$$\rightarrow \varepsilon_{x'} = \frac{ds' - ds}{ds} = \varepsilon_x \frac{dx^2}{ds^2} = \varepsilon_x \cos^2\alpha$$

$$\theta_{x'} = \varepsilon_x dx \sin\alpha / ds = \varepsilon_x \sin\alpha \cos\alpha$$

x'轴顺时针转动角度.

• ε_y 对转角α处线段的影响

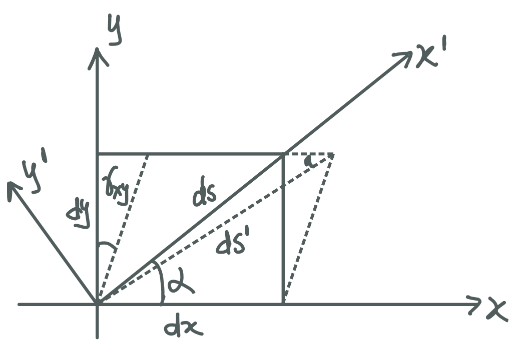


$$\varepsilon_{x'} = \varepsilon_y \left(\frac{dy}{ds}\right)^2 = \varepsilon_y \sin^2\alpha$$

$$\theta_{x'} = -\varepsilon_y \sin\alpha \cos\alpha$$

逆时针

• γ_{xy} 对转角α处线段的影响



$$ds' = \sqrt{(dx + \gamma_{xy} dy)^2 + dy^2} = \sqrt{dx^2 + dy^2 + 2\gamma_{xy} dx\, dy}$$

$$\varepsilon_{x'} = \frac{ds' - ds}{ds} = \gamma_{xy} \frac{dx\, dy}{ds^2} = \gamma_{xy} \sin\alpha \cos\alpha$$

$$\theta_{x'} = \frac{\gamma_{xy}\, dy \sin\alpha}{ds} = \gamma_{xy} \sin^2\alpha$$

- $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ 对转角 α 处（x'方向）线段的综合影响

$$\boxed{\begin{aligned}\varepsilon_{x'} &= \varepsilon_x \cos^2\alpha + \varepsilon_y \sin^2\alpha + \gamma_{xy}\sin\alpha\cos\alpha \\ &= \frac{\varepsilon_x+\varepsilon_y}{2} + \frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\alpha + \frac{\gamma_{xy}}{2}\sin 2\alpha\end{aligned}}$$

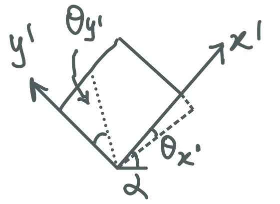
注：交叉项为高阶项，小变形下可忽略

$$\theta_{x'} = \varepsilon_x \sin\alpha\cos\alpha - \varepsilon_y \sin\alpha\cos\alpha + \gamma_{xy}\sin^2\alpha \quad (x'\text{轴}\circlearrowleft)$$

- $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ 对转角 $\alpha+\frac{\pi}{2}$ 处（y'方向）线段的综合影响

$$\boxed{\varepsilon_{y'} = \frac{\varepsilon_x+\varepsilon_y}{2} - \frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\alpha - \frac{\gamma_{xy}}{2}\sin 2\alpha}$$

$$\theta_{y'} = -\varepsilon_x \sin\alpha\cos\alpha + \varepsilon_y \sin\alpha\cos\alpha + \gamma_{xy}\cos^2\alpha \quad (y'\text{轴}\circlearrowleft)$$



$$\gamma_{x'y'} = \theta_{y'} - \theta_{x'} = -2(\varepsilon_x - \varepsilon_y)\sin\alpha\cos\alpha + \gamma_{xy}(\cos^2\alpha - \sin^2\alpha)$$

$$\rightarrow \boxed{\frac{1}{2}\gamma_{x'y'} = -\frac{\varepsilon_x-\varepsilon_y}{2}\sin 2\alpha + \frac{\gamma_{xy}}{2}\cos 2\alpha}$$

$\rightarrow$ E 在不同坐标系下的分量满足 $E' = AEA^T$（与 Σ 行为相同）：

$$\begin{bmatrix} \varepsilon_{x'} & \frac{1}{2}\gamma_{y'x'} \\ \frac{1}{2}\gamma_{x'y'} & \varepsilon_{y'} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{yx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

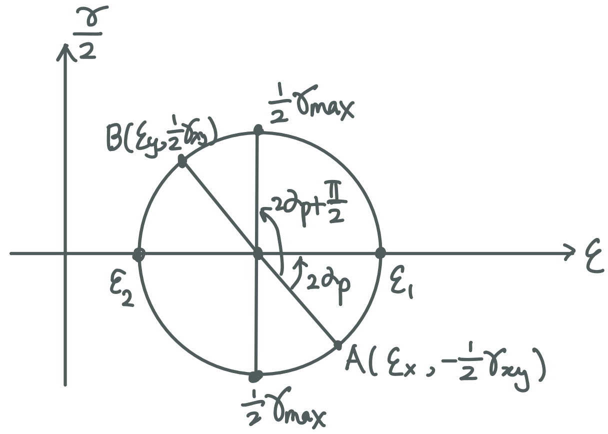
因此，可以采用关于 Σ 的其它结论（注意 $\tau_{xy} \leftrightarrow \frac{1}{2}\gamma_{xy}$ 替换）.

- $\varepsilon_{x'} + \varepsilon_{y'} = \varepsilon_x + \varepsilon_y$

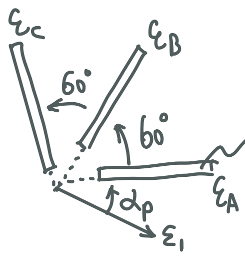
- 主应变 $\varepsilon_{1,2} = \dfrac{\varepsilon_x+\varepsilon_y}{2} \pm \sqrt{\left(\dfrac{\varepsilon_x-\varepsilon_y}{2}\right)^2 + \left(\dfrac{\gamma_{xy}}{2}\right)^2}$，主应变方向 $2\alpha_p = \dfrac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$

· 最大切应变 $\frac{1}{2}\gamma_{max}=\sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2+\left(\frac{\gamma_{xy}}{2}\right)^2}$ ， $2\alpha_s=2\alpha_p+\frac{\pi}{2}$ 或 $2\alpha_p+\frac{3\pi}{2}$

· 应变莫尔圆 $x_c=\frac{\varepsilon_x+\varepsilon_y}{2}$ ， $R=\sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2+\left(\frac{\gamma_{xy}}{2}\right)^2}$



值得注意E为可实验测量的张量，通常采用等角应变花或直角应变花测量.



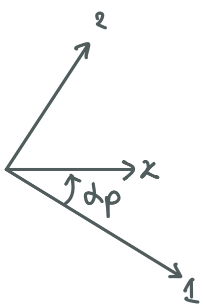
$\underbrace{\text{线应变}\to\text{电阻变化}}_{\text{测量}}$

等角应变花

$$\underbrace{\begin{bmatrix}\varepsilon_A\\ \varepsilon_B\\ \varepsilon_c\end{bmatrix}\to\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \frac{1}{2}\gamma_{xy}\end{bmatrix}}_{\uparrow\ \text{作业题 4.7}}\ \text{或}\ \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \alpha_p\end{bmatrix}$$

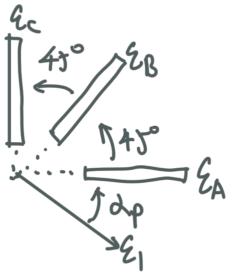
$$\begin{cases}\varepsilon_{x'}=\frac{\varepsilon_x+\varepsilon_y}{2}+\frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\alpha+\frac{\gamma_{xy}}{2}\sin 2\alpha\\ \alpha=0,\ \varepsilon_{x'}=\varepsilon_A;\quad \alpha=60°,\ \varepsilon_{x'}=\varepsilon_B;\quad \alpha=120°,\ \varepsilon_{x'}=\varepsilon_B\end{cases}$$

如何直接测出主应变 $\varepsilon_1,\varepsilon_2$ 以及主方向 α_p？



在主坐标系下，$E=\begin{bmatrix}\varepsilon_1 & 0\\ 0 & \varepsilon_2\end{bmatrix}$，但不清楚主轴与x轴夹角，设为 α_p

$$\left.\begin{aligned}\varepsilon_A&=\frac{\varepsilon_1+\varepsilon_2}{2}+\frac{\varepsilon_1-\varepsilon_2}{2}\cos 2\alpha_p\\ \varepsilon_B&=\frac{\varepsilon_1+\varepsilon_2}{2}+\frac{\varepsilon_1-\varepsilon_2}{2}\cos 2(\alpha_p+60°)\\ \varepsilon_c&=\frac{\varepsilon_1+\varepsilon_2}{2}+\frac{\varepsilon_1-\varepsilon_2}{2}\cos 2(\alpha_p+120°)\end{aligned}\right\}\to\begin{cases}2\alpha_p=\arctan\frac{\sqrt{3}(\varepsilon_c-\varepsilon_B)}{2\varepsilon_A-\varepsilon_B-\varepsilon_c}\\ \varepsilon_{1,2}=\frac{1}{3}(\varepsilon_A+\varepsilon_B+\varepsilon_c)\pm\sqrt{\left(\frac{\varepsilon_B+\varepsilon_c-2\varepsilon_A}{3}\right)^2+\left(\frac{\varepsilon_c-\varepsilon_B}{\sqrt{3}}\right)^2}\end{cases}$$

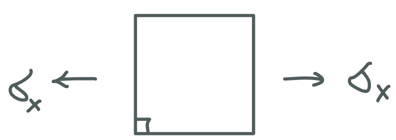


直角应变花原理相同→

$$\begin{cases} 2\alpha_p = \arctan \dfrac{2\varepsilon_B - \varepsilon_A - \varepsilon_C}{\varepsilon_A - \varepsilon_C} \\ \varepsilon_{1,2} = \dfrac{\varepsilon_A + \varepsilon_C}{2} \pm \sqrt{\left(\dfrac{\varepsilon_A - \varepsilon_C}{2}\right)^2 + \left(\dfrac{2\varepsilon_B - \varepsilon_A - \varepsilon_C}{2}\right)^2} \end{cases}$$

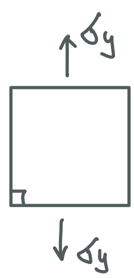
§7.6. 广义胡克定律

我们在之前的教学中，已经定义了材料抵抗变形的能力，特别是平面应力状态下



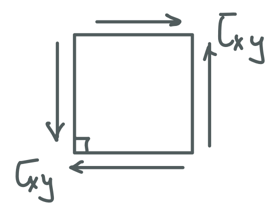
$\varepsilon_x = \dfrac{\sigma_x}{E}$, $\varepsilon_y = -\nu \dfrac{\sigma_x}{E}$

$\gamma_{xy} = 0$



$\varepsilon_x = -\nu \dfrac{\sigma_y}{E}$, $\varepsilon_y = \dfrac{\sigma_y}{E}$

$\gamma_{xy} = 0$



$\varepsilon_x = 0$, $\varepsilon_y = 0$, $\gamma_{xy} = \dfrac{\tau_{xy}}{G}$

→ 正、切应力引起的应变彼此无关

(小变形下)

在更一般形式的$\underset{2\times 2}{\Sigma}$下的E可表示为

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}(\sigma_x - \nu\sigma_y) \\ \varepsilon_y = \dfrac{1}{E}(\sigma_y - \nu\sigma_y) \\ \gamma_{xy} = \dfrac{\tau_{xy}}{G} \end{cases}$$

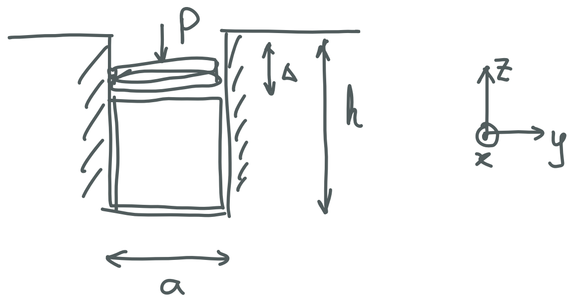
(稍后证明 $G = \dfrac{E}{2(1+\nu)}$)

这可进一步推广到3x3的Σ−E关系

广义胡克定律

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z) \\ \varepsilon_y = \dfrac{1}{E}(\sigma_y - \nu\sigma_x - \nu\sigma_y) \\ \varepsilon_z = \dfrac{1}{E}(\sigma_z - \nu\sigma_x - \nu\sigma_y) \\ \gamma_{xy} = \dfrac{1}{G}\tau_{xy}, \quad \gamma_{yz} = \dfrac{1}{G}\tau_{yz}, \quad \gamma_{xz} = \dfrac{1}{G}\tau_{xz} \end{cases}$$

例:



P-Δ关系?

应变状态: $\varepsilon_z = -\dfrac{\Delta}{h}$, $\varepsilon_x = \varepsilon_y = \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0$

$$\left.\begin{aligned}\varepsilon_x &= \frac{1}{E}\sigma_x - \frac{\nu}{E}(\sigma_y + \sigma_z)\\ \varepsilon_y &= \frac{1}{E}\sigma_y - \frac{\nu}{E}(\sigma_x + \sigma_z)\\ \varepsilon_z &= \frac{1}{E}\sigma_z - \frac{\nu}{E}(\sigma_x + \sigma_y)\end{aligned}\right. \Rightarrow \begin{cases}\sigma_x = \sigma_y = -\dfrac{E\nu}{(1+\nu)(1-2\nu)}\dfrac{\Delta}{h} = -\underbrace{\lambda}_{\text{Lame}}\dfrac{\Delta}{h}\\[2ex] \sigma_z = -\dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}\dfrac{\Delta}{h} = -\dfrac{P}{a^2}\end{cases}$$

$$\rightarrow P = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{a^2}{h}\Delta \qquad \begin{cases}\cdot\text{当 } \nu = 0.5 \text{ 时, } k = P/\Delta \rightarrow \infty\\ \cdot\text{当 } \nu = 0 \text{ 时, } P = EA\dfrac{\Delta}{h} = EA\varepsilon\end{cases}$$

• 体积模量

切应变不会带来微元体积的变化，正应变带来的体积相对变化为

$$\varepsilon_V = \frac{\Delta V}{V} = \frac{(1+\varepsilon_x)dx\,(1+\varepsilon_y)dy\,(1+\varepsilon_z)dz - dx\,dy\,dz}{dx\,dy\,dz} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

定义球应力 $p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \sigma_8$ (静水压下 当 $\sigma_1 = \sigma_2 = \sigma_3 = -p_0$, $p = -p_0$)

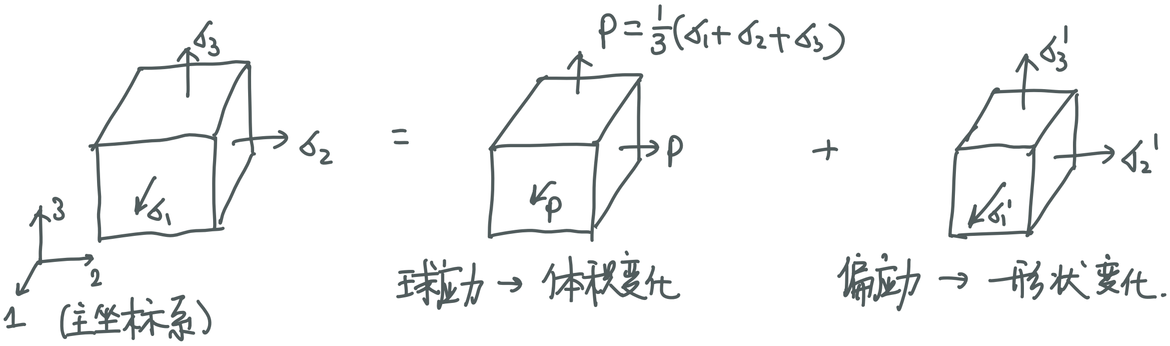
$$\varepsilon_V = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3) + \frac{1}{E}(\sigma_2 - \nu\sigma_1 - \nu\sigma_3) + \frac{1}{E}(\sigma_3 - \nu\sigma_1 - \nu\sigma_2)$$

$$= \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3)$$

体积模量定义 $K=\frac{p}{\varepsilon_v}=\frac{E}{3(1-2\nu)}$

静水压下，$p=-P_0$，$\varepsilon_v=-\frac{P_0}{K}=-\frac{3P_0(1-2\nu)}{E}\le 0 \to \nu<\frac{1}{2}$ （$\underbrace{\nu=\frac{1}{2}}_{\to K\to\infty}$ 代表不可压缩）

• 应力偏量和应变偏量



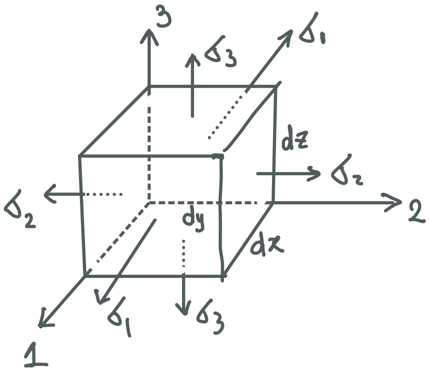
应力偏量：$\sigma_1'=\overbrace{\sigma_1-p}^{=\frac{1}{3}(2\sigma_1-\sigma_2-\sigma_3)}$，$\sigma_2'=\sigma_2-p$，$\sigma_3'=\sigma_3-p$，$\sigma_1'+\sigma_2'+\sigma_3'=0$

应变偏量：$\varepsilon_1'=\varepsilon_1-\frac{1}{3}\varepsilon_v$，$\varepsilon_2'=\varepsilon_2-\frac{1}{3}\varepsilon_v$，$\varepsilon_3'=\varepsilon_3-\frac{1}{3}\varepsilon_v$，$\varepsilon_1'+\varepsilon_2'+\varepsilon_3'=0$

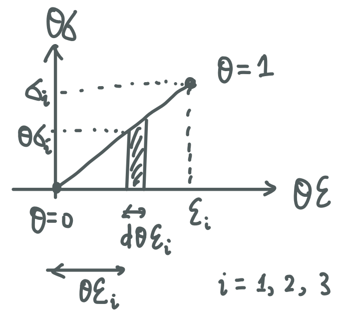
$$\varepsilon_1'=\frac{1}{E}(\sigma_1-\nu\sigma_2-\nu\sigma_3)-\frac{1-2\nu}{3E}(\sigma_1+\sigma_2+\sigma_3)=\frac{(1+\nu)}{3E}(2\sigma_1-\sigma_2-\sigma_3)=\underbrace{\frac{1+\nu}{E}}_{\nu\ge-1,\ \nu=-1\text{不可剪切}}\sigma_1'=\frac{\sigma_1'}{2G}\to\boxed{\varepsilon_i'=\frac{\sigma_i'}{2G}}$$

§7.7 弹性应变能

对于复杂应力状态，先寻找主平面方向，建立主坐标系，分析主应力做的功（单位体积）



施加 $\theta\sigma_1$，$\theta\sigma_2$，$\theta\sigma_3$，产生 $\theta\varepsilon_1$，$\theta\varepsilon_2$，$\theta\varepsilon_3$



$$du = \frac{\overbrace{\theta\sigma_1 \times dy\,dz}^{\text{合力}} \cdot \overbrace{\varepsilon_1 d\theta \cdot dx}^{\text{位移}}}{\underbrace{dx\,dy\,dz}_{\text{单位体积}}} + \theta\sigma_2\varepsilon_2 d\theta + \theta\sigma_3\varepsilon_3 d\theta$$

应变比能 $u = \int du = \int_0^1 \sum_{i=1}^{3} \theta\sigma_i\varepsilon_i d\theta = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3)$

注：不一定假设应力分量按同一参数成比例变化，弹性体最终状态与力作用的过程无关（弹性力学）

我们选择比例加载是为了简化计算.

广义胡克 → $u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$

可以分解为 $u = \underbrace{\frac{1+\nu}{6E}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]}_{u_s} + \underbrace{\frac{1-2\nu}{6E}(\sigma_1+\sigma_2+\sigma_3)^2}_{u_v}$

其物理含义为 $u_s = \frac{3\tau_8^2}{4G} = \frac{1}{2}(\sigma_1'\varepsilon_1' + \sigma_2'\varepsilon_2' + \sigma_3'\varepsilon_3')$ 形状改变比能

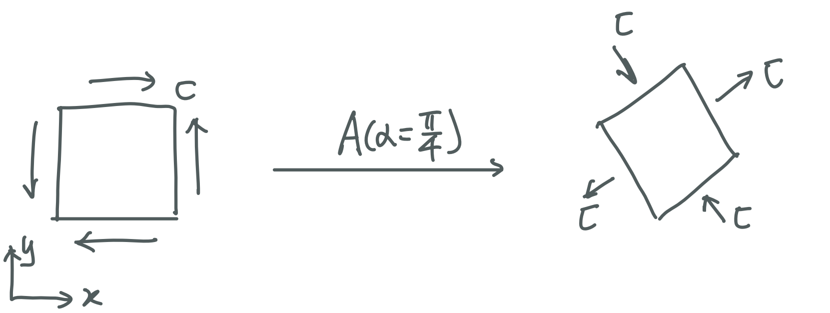
$u_v = \frac{\sigma_8^2}{2K} = \frac{p^2}{2K} = \frac{1}{2}p\varepsilon_v$ 体积改变比能

注意 $\sigma_i=\sigma_i'+p$，$\varepsilon_i=\varepsilon_i'+\frac{1}{3}\varepsilon_v$，应力、应变可以叠加，能量通常不能。上述分解成立的原因是偏应力不对球应变做功 & 球应力不对偏应变做功。形状与体积改变可解耦.

$$u=\frac{1}{2}\left[(\sigma_1'+p)(\varepsilon_1'+\tfrac{1}{3}\varepsilon_v)+(\sigma_2'+p)(\varepsilon_2'+\tfrac{1}{3}\varepsilon_v)+(\sigma_3'+p)(\varepsilon_3'+\tfrac{1}{3}\varepsilon_v)\right]$$

$$=\underbrace{\frac{1}{2}(\sigma_1'\varepsilon_1'+\sigma_2'\varepsilon_2'+\sigma_3'\varepsilon_3')}_{u_s}+\underbrace{\frac{1}{2}p\varepsilon_v}_{u_v}+\frac{1}{6}\varepsilon_v\cancelto{0}{(\sigma_1'+\sigma_2'+\sigma_3')}+\frac{1}{2}p\cancelto{0}{(\varepsilon_1'+\varepsilon_2'+\varepsilon_3')}$$

例：证明 $G=\frac{E}{2(1+\nu)}$



$\sigma_1=\tau$，$\sigma_2=-\tau$，$\sigma_3=0$

在第三章证明了 $u=\underbrace{\frac{1}{2G}}\tau^2$　　在主坐标系下 $u=\underbrace{\frac{1}{2E}(\tau^2+\tau^2+2\nu\tau^2)}$

$\rightarrow G=\frac{E}{2(1+\nu)}$

更一般的平面应力状态 $\sigma_{1,2}=\frac{\sigma_x+\sigma_y}{2}\pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$，$\sigma_3=0$

$$\rightarrow\boxed{u=\frac{1}{2E}\left[(\sigma_x+\sigma_y)^2+2(1+\nu)(\tau_{xy}^2-\sigma_x\sigma_y)\right]}$$

§7.8. 强度理论

- 第一类强度理论（脆性破坏）

① 第一强度理论——最大拉应力理论

破坏条件：$\sigma_1 = \sigma_b$ ← 在单轴拉伸（$\sigma_1=\sigma,\ \sigma_2=\sigma_3=0$）下标定

任意应力状态

强度条件：$\sigma_1 \le [\sigma]$，$[\sigma] = \dfrac{\sigma_b}{n_b}$

② 第二强度理论——最大伸长应变理论

破坏条件：$\varepsilon_1 = \varepsilon_b$ ← 可在单轴拉伸下标定

任意状态

$$\sigma_1 - \nu(\sigma_2+\sigma_3) = E\varepsilon_b = \sigma_b$$ ← 单轴拉伸下的最大拉应力

强度条件：$\sigma_1 - \nu(\sigma_2+\sigma_3) \le [\sigma]$，$[\sigma] = \dfrac{\sigma_b}{n_b}$

实验表明当（$\sigma_3=0$）
$$\begin{cases} \sigma_1,\ \sigma_2 > 0 \text{ 时，采用第一强度理论} \\ \sigma_1>0,\ \sigma_2<0,\ |\sigma_1|>|\sigma_2| \text{ 时，第一} \\ \sigma_1>0,\ \sigma_2<0,\ |\sigma_1|<|\sigma_2| \text{ 时，第二} \end{cases}$$

• 第二类强度理论（出现屈服或发生塑性变形破坏）

③ 第三强度理论 — 最大切应力理论

当外力过大时，构件上的危险点处的材料会沿最大切应力所在平面滑移，屈服破坏

破坏条件：$\underbrace{\tau_{max} = \frac{\sigma_1-\sigma_3}{2}}_{\text{任意状态}} = \underbrace{\tau_s}_{\text{可在单轴拉伸下}(\sigma_1=\sigma,\ \sigma_2=\sigma_3=0)\text{标定}} = \frac{\sigma_s}{2}$

Tresca屈服准则：$\sigma_1 - \sigma_3 = \sigma_s$

强度条件：$\sigma_1 - \sigma_3 \le [\sigma_s]$，$[\sigma_s] = \frac{\sigma_s}{n_s}$

④ 第四强度理论 — 最大形状改变能理论

单轴试验标定

破坏条件：$u_s = \frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] = u_{ss} = \frac{1}{6G}\sigma_s^2$

Von Mises屈服准则：$\sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]} = \sigma_s$

强度条件：$\sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]} \le [\sigma]$，$[\sigma] = \frac{\sigma_s}{n_s}$

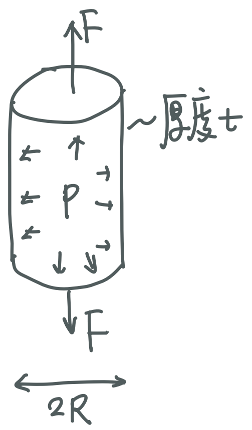
• 许用正应力与许用切应力关系（以纯剪切为例 $\sigma_1=\tau$，$\sigma_2=0$，$\sigma_3=-\tau$）

Tresca（第三）：$\tau-(-\tau) \le [\sigma] \rightarrow [\tau] = 0.5[\sigma]$

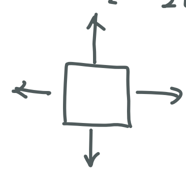
Von Mises（第四）：$\sqrt{\frac{1}{2}(\tau^2+\tau^2+4\tau^2)} \le [\sigma] \rightarrow [\tau] = \frac{1}{\sqrt{3}}[\sigma] = 0.577[\sigma]$

• 关于第三、四强度理论的试验（平面应力状态，$\sigma_3=0$）

试验①：轴力、压力作用的薄壁圆管



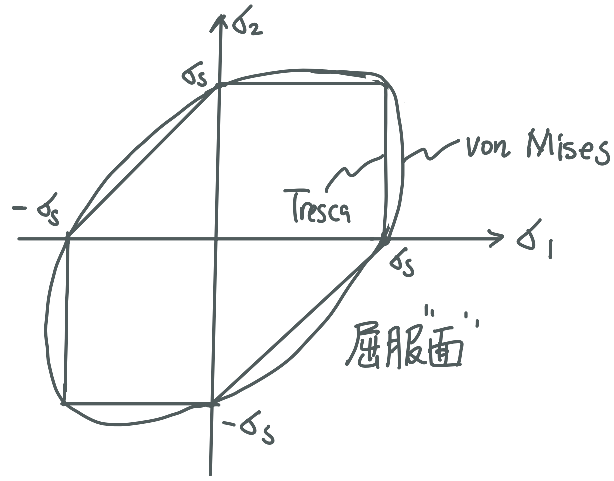
$$\sigma_2=\frac{PR}{2t}+\frac{F}{2\pi Rt}$$



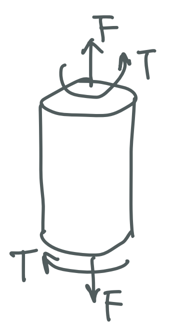
$$\sigma_1=\frac{PR}{t}\quad,\ \sigma_3=0$$

Tresca屈服准则：$\mathrm{Max}\{|\sigma_1-\sigma_2|,|\sigma_1|,|\sigma_2|\}=\sigma_s$

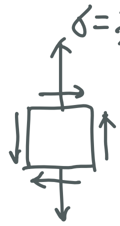
Von Mises屈服准则：$\sqrt{\frac{1}{2}[(\sigma_1-\sigma_2)^2+\sigma_1^2+\sigma_2^2]}=\sigma_s$



试验②：拉伸、扭转作用下的薄壁圆管



$$\sigma=\frac{F}{2\pi Rt}$$

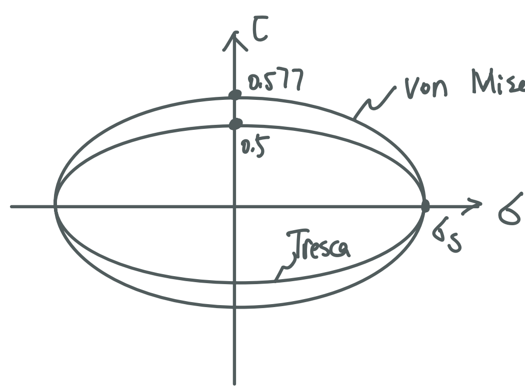


$$\tau=\frac{T}{2\pi R^2t}$$

$$\sigma_{1,2}=\frac{\sigma}{2}\pm\sqrt{\left(\frac{\sigma}{2}\right)^2+\tau^2},\quad \sigma_3=0$$

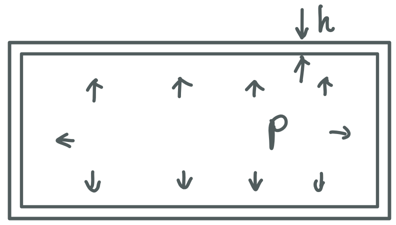
von Mises：$\sigma^2+3\tau^2=\sigma_s^2$

Tresca：$\sigma^2+4\tau^2=\sigma_s^2$



☆大部分金属十分接近第四（von Mises）强度理论

例：



$p = 1.5\ \mathrm{MPa}$，$R = 600\ \mathrm{mm}$

$[\sigma] = 170\ \mathrm{MPa}$，$\nu = 0.28$.

求 h？

$$\sigma_1 = \frac{pR}{t}, \quad \sigma_2 = \frac{pR}{2t}, \quad \sigma_3 = -p$$

最大拉应力：$\sigma_1 = [\sigma] \rightarrow t = \frac{pR}{[\sigma]} = 5.3\ \mathrm{mm}$

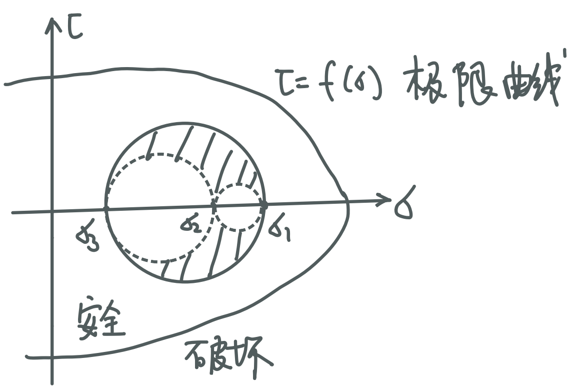
最大拉应变：$\sigma_1 - \nu(\sigma_2 + \sigma_3) = [\sigma] \rightarrow t = 4.6\ \mathrm{mm}$

最大切应力：$\sigma_1 - \sigma_3 = \frac{pR}{t} + p = [\sigma] \rightarrow t = 5.4\ \mathrm{mm}$

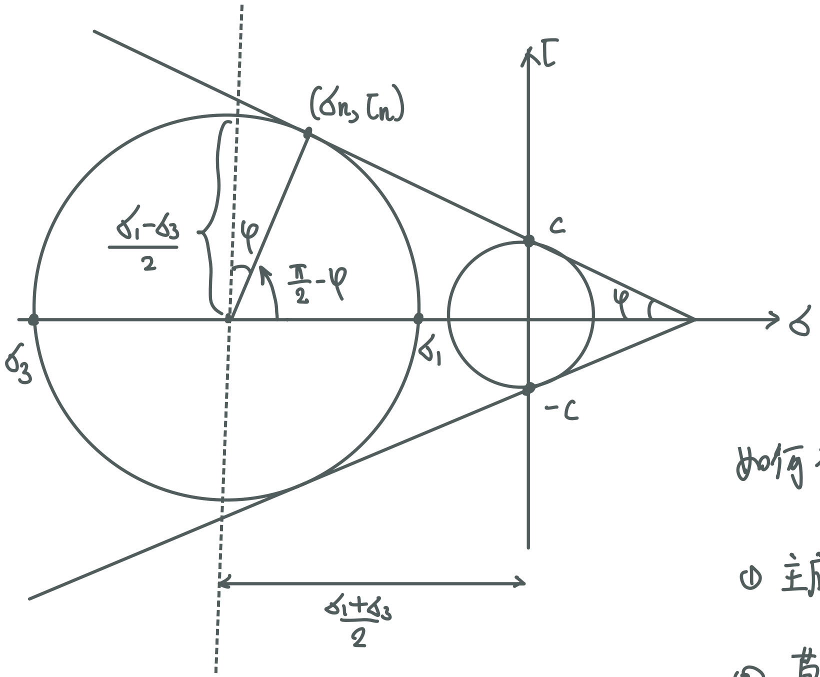
最大形状改变比能：$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2[\sigma]^2 \rightarrow t = 4.6\ \mathrm{mm}$

- 莫尔强度理论

在物体内一点的某个截面上，当其正应力与切应力达到某种最不利的组合时，产生破坏。



莫尔库仑强度理论：$|\tau_n| = \underset{\text{粘聚力}}{c} - \underbrace{\sigma_n \tan\varphi}_{\text{内摩擦阻力}}$ ← 内摩擦角



岩石、混凝土等抗压能力远大于抗拉能力

如何确定σ_n, τ_n与σ_1, σ_3关系?

① 主应力状态+转角公式 ($\theta = \frac{\pi}{4} - \frac{\varphi}{2}$)

② 莫尔圆关系

$$\sigma_n = \frac{\sigma_1+\sigma_3}{2} + \frac{\sigma_1-\sigma_3}{2}\sin\varphi$$

$$\tau_n = \frac{\sigma_1-\sigma_3}{2}\cos\varphi$$

代入莫尔库仑强度理论:

$$\frac{\sigma_1}{\underbrace{\left(\frac{2c\cos\varphi}{1-\sin\varphi}\right)}_{\sigma_t}} - \frac{\sigma_3}{\underbrace{\left(\frac{2c\cos\varphi}{1+\sin\varphi}\right)}_{\sigma_c}} = 1$$

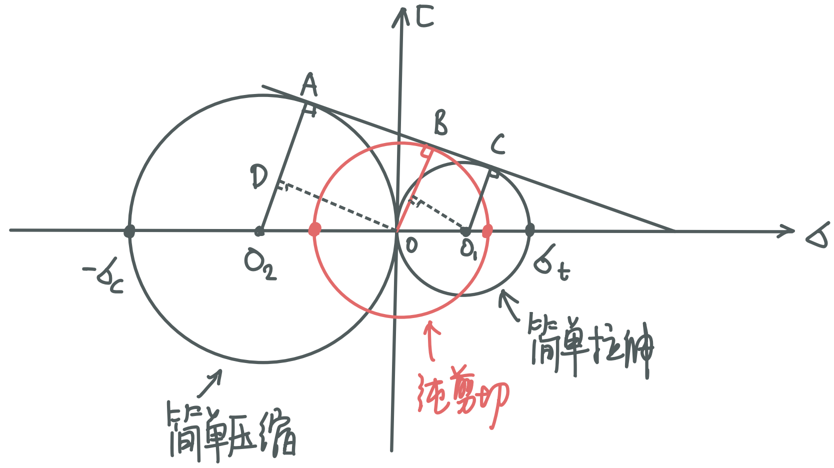
	c	φ
砂岩	27.2	37.8
大理岩	21.2	25.3
花岗岩	55.1	51.0

$\rightarrow \frac{\sigma_1}{\sigma_t} - \frac{\sigma_3}{\sigma_c} = 1$ 或 $\sigma_1 - m\sigma_3 = 1$, $m = \frac{\sigma_t}{\sigma_c} \sim 0.2-0.3$ (铸铁)

为什么这样定义? 单轴拉伸 $\sigma_1=\sigma$, $\sigma_3=0 \rightarrow \sigma_1=\sigma_t$ 时拉伸破坏.

单轴压缩 $\sigma_1=0$, $\sigma_3=-\sigma \rightarrow \sigma=\sigma_c$ 时压缩破坏.

例: 铸铁拉伸强度σ_t, 压缩强度σ_c满足$\sigma_t = m\sigma_c$, 用莫尔-库仑强度理论求剪切强度τ_s.



$$\frac{O_2A - OB}{OO_2} = \frac{OB - O_1C}{OO_1}$$

$$\rightarrow \frac{\frac{1}{2}\sigma_c - \tau_s}{\frac{1}{2}\sigma_c} = \frac{\tau_s - \frac{1}{2}\sigma_t}{\frac{1}{2}\sigma_t}$$

$$\sigma_t = m\,\sigma_c$$

$$\rightarrow \tau_s = \frac{m}{1+m}\sigma_c = \frac{1}{1+m}\sigma_t$$