北京大学
PEKING UNIVERSITY

材料力学 (Mechanics of Materials)

压杆稳定性问题

戴兆贺

北京大学工学院

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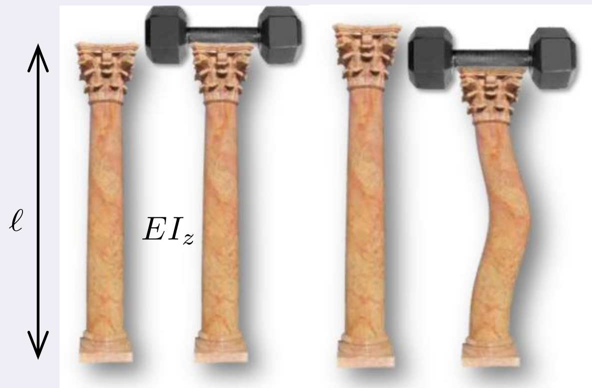
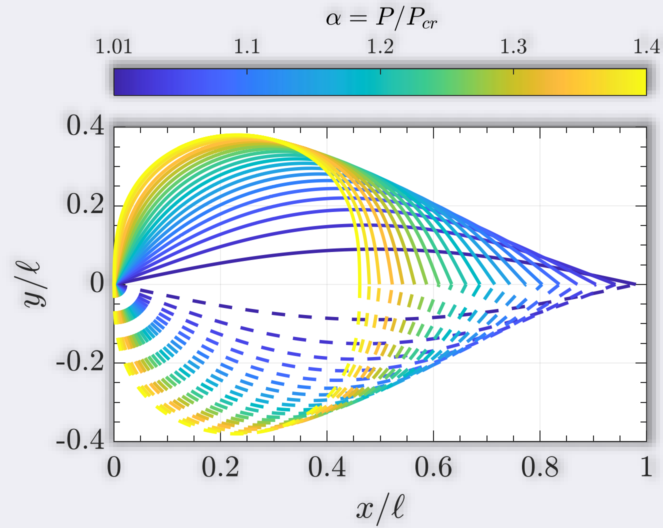
欧拉临界力



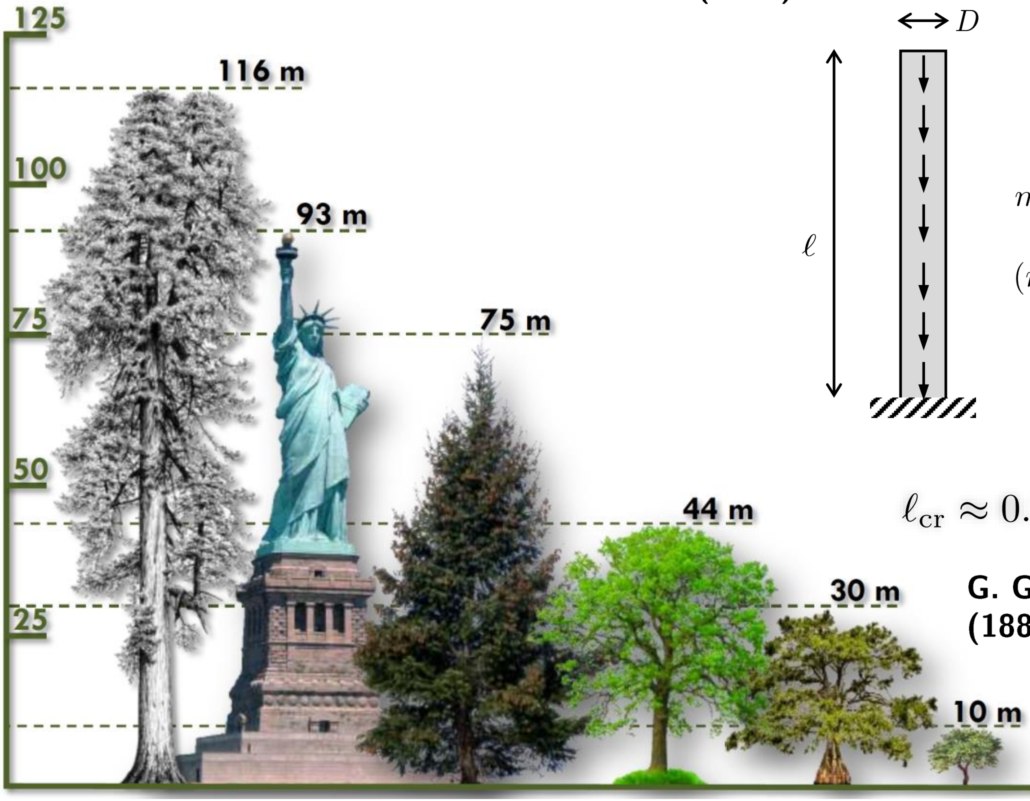
$$\mathcal{P}_{\mathrm{cr}} = \frac{\pi^2 EI_z}{(\mu\ell)^2}$$



- **长度折算系数：**具体数值依赖于具体的载荷情况和约束条件
- **线性理论：**$P = P_{\mathrm{cr}}$ 后，细长杆的最大挠度为任意值
- **非线性理论：**$P > P_{\mathrm{cr}}$ 后，存在“两个”稳定解答

Size and Shape in Biology

T. McMahon, Science (1973)



$$mg \sim \rho g \times D^2 \times \ell$$

$$(mg)_{\mathrm{cr}} \sim \frac{EI_z}{(\mu\ell)^2}$$

$$\ell_{\mathrm{cr}} \approx 0.8 \left(\frac{E}{\rho}\right)^{1/3} D^{2/3}$$

G. Greenhill (1881)

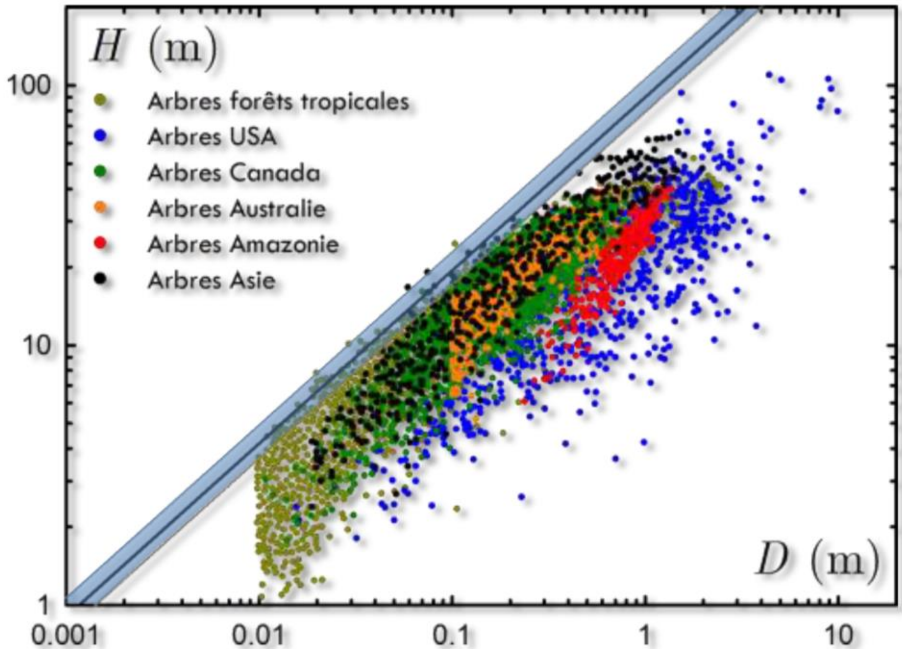
The Tesla tree?

树木能长多高?

$$10 < E < 12 \text{ GPa}$$

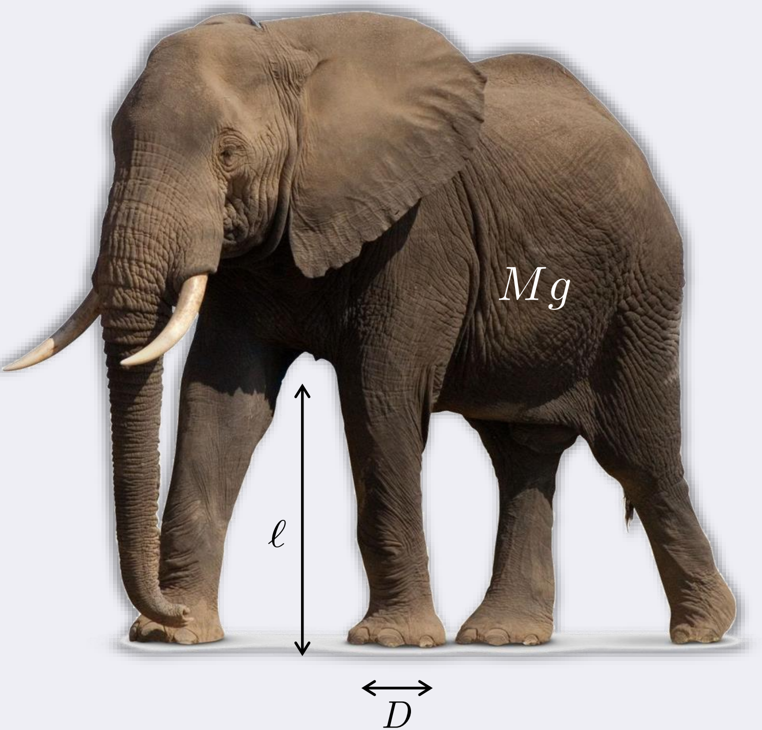
$$500 < \rho < 1000 \text{ kg/m}^2$$

$$\ell_{cr} \approx 0.8 \left(\frac{E}{\rho}\right)^{1/3} D^{2/3}$$



Fabian Brau (2013)

动物骨骼的尺寸和形状?



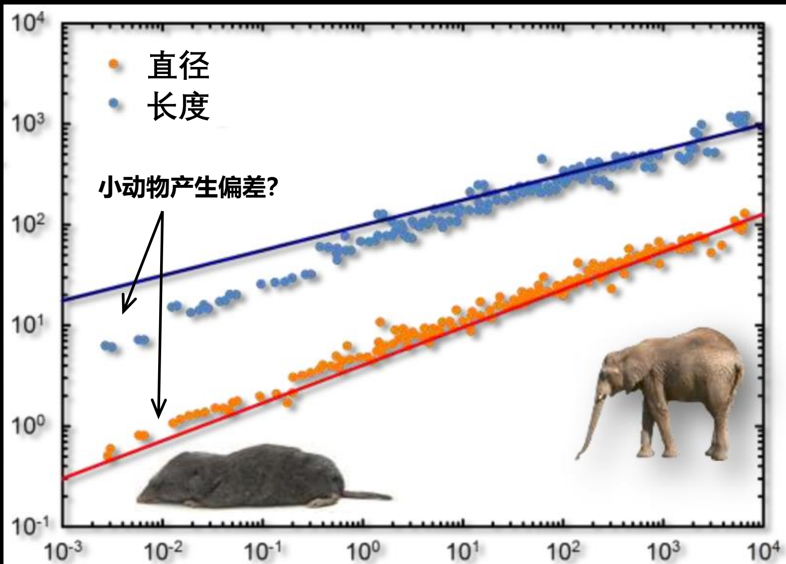
失稳准则： $Mg \sim \frac{ED^4}{\ell^2}$

骨骼/体重关系： $\rho D^2 \ell \sim \phi M$

$$D \sim \phi^{1/4} \left(\frac{g}{E\rho^2}\right)^{1/8} M^{3/8}$$

$$\ell \sim \phi^{1/2} \left(\frac{E}{g\rho^2}\right)^{1/4} M^{1/4}$$

标度关系



长度（mm）

重量（kg）

Elephant

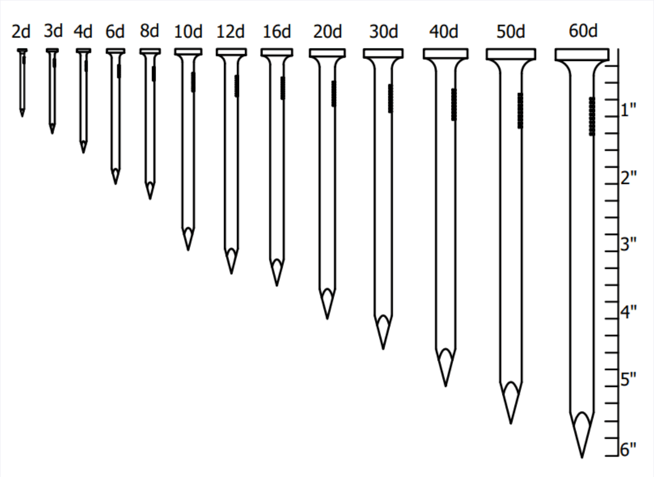
$\phi \sim 20\%$

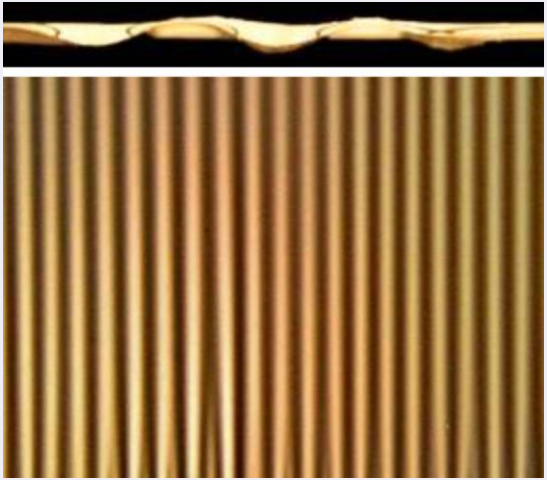
Rat

$\phi \sim 5\%$

“强度准则”导致显著不同的骨骼/体重比
(Square–cube law)

其他案例

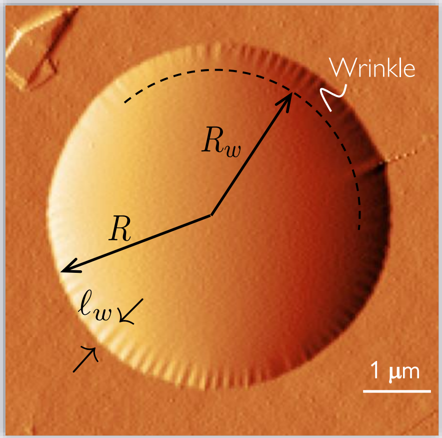




$$\ell \propto D^{3/2}$$

$$\lambda \sim (EI_z/\rho g)^{1/4}$$

2D problem: A far from the threshold problem



❑ First consider a thick solid (near-threshold approach):

$$w(r,\theta) = \widetilde{w}(r) + w^{(1)}(r)\cos m\theta + \ldots$$

$$N_{ij}(r,\theta) = \widetilde{N}_{ij}(r) + N_{ij}^{(1)}(r)\cos m\theta + \ldots$$

To solve eigenvalue problem with $\widetilde{f} \gg f^{(1)}$ (cf. Euler buckling)

The base

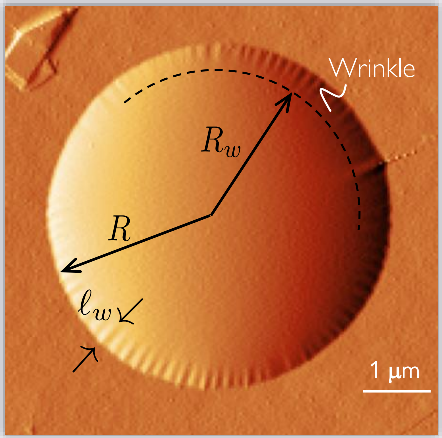
❑ But for an ultrathin solid, the state is far from threshold:

$$m \to \infty \text{ as } t \to 0,\ \mathcal{U}_b/\mathcal{U}_s \to 0 \quad \text{(many wrinkles in 3D)}$$

$$m \times w^{(1)}(r) = \mathcal{O}(1) \text{ so that } (\partial w/\partial\theta)^2 = \mathcal{O}(1)$$

Dai et al. J. Mech. Phys. Solids (2021); Dai et al. Int. J. Solids Struct. (2022)

Tension field theory



❑ What happened in the wrinkled region? $(R_w < r \leq R)$

$$w(r,\theta) = \widetilde{w}(r) + \frac{w^{(1)}(r)}{m}\cos m\theta + \ldots$$

$$N_{ij}(r,\theta) = \widetilde{N}_{ij}(r) + \frac{N_{ij}^{(1)}(r)}{m}\cos m\theta + \ldots$$

The base

$$\Rightarrow \widetilde{N}_{\theta\theta} = N_{\theta\theta}^{(1)} = 0$$

Only to solve the mean (axisymmetric) shape with $N_{rr} \gg N_{\theta\theta}$

The hoop stress state is relieved completely by wrinkling: Equivalent to $h/R \gg \mathcal{K}^{-1/2}$

Dai et al. J. Mech. Phys. Solids (2021); Dai et al. Int. J. Solids Struct. (2022)

Thanks!