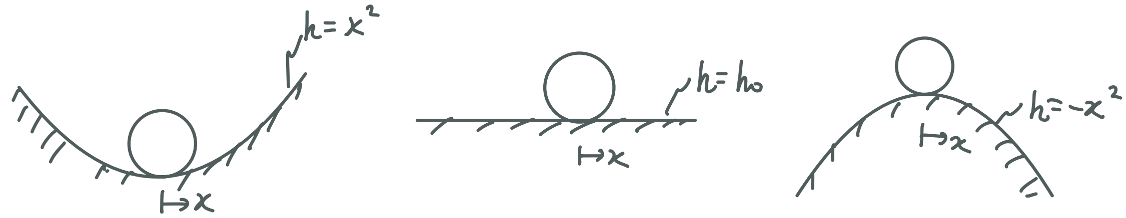
第六章 压杆稳定性

§6.1. 稳定性问题的提法

考虑势能 $\Pi=\Pi(x)$，$\frac{\partial \Pi}{\partial x}=0$ 可给出平衡位置（注意变分原理 $\underbrace{\frac{\delta \Pi}{\delta f(x)}}_{\text{弹性力学}}=0$ 可以给出平衡方程），但该位置的稳定性需考查 $\frac{\partial^2 \Pi}{\partial x^2}$.

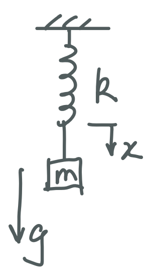
势能

$$\Pi = mgh(x)+C$$



$\frac{\partial \Pi}{\partial x}=0 \to x=0$	$\frac{\partial \Pi}{\partial x}=0$	$\frac{\partial \Pi}{\partial x}=0 \to x=0$
$\frac{\partial^2 \Pi}{\partial x^2}>0 \to x=0$为极小值点（稳定）	$\frac{\partial^2 \Pi}{\partial x^2}=0$	$\frac{\partial^2 \Pi}{\partial x^2}<0 \to$ 极大值点.（不稳定）

考虑一个简单、存在弹性变形能的系统的势能：

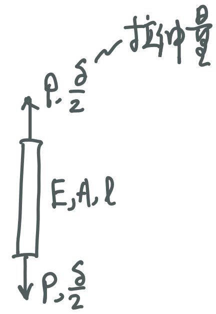


$$\Pi=\frac{1}{2}k\underbrace{x}_{\text{弹簧变形量}}^2-mg\underbrace{x}_{}$$

同样地 $\Longrightarrow$

$\frac{\partial \Pi}{\partial x}=0 \to kx=mg$（平衡解）

$\frac{\partial^2 \Pi}{\partial x^2}=k>0 \to$ 该解是"稳定"

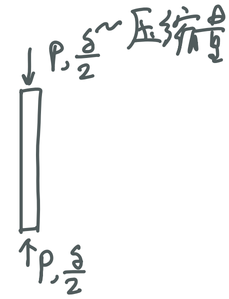


$$\Pi=\underbrace{\frac{1}{2}E\varepsilon^2 V}_{\text{弹性势能}}-\underbrace{P\delta}_{\text{外力势能}}$$

$$=\frac{1}{2}\frac{EA}{l}\delta^2-P\delta$$

$\frac{\partial \Pi}{\partial \delta}=0 \to \delta=\frac{Pl}{EA}$

$\frac{\partial^2 \Pi}{\partial \delta^2}=\frac{EA}{l}>0$ "稳定"



$$\Pi=\frac{1}{2}\frac{EA}{l}\delta^2-P\delta$$

$\to \delta=\frac{Pl}{EA}$ 且稳定

但是实际情并非如此!

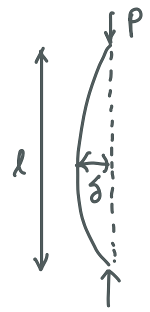
生活经验：P为压时，杆件并不稳定. 为什么?

答案：压杆失稳的变形模式并非压缩，而是弯曲！——重新考虑含弯曲变形能的总势能

$$\Pi = \underbrace{\int_0^l \frac{1}{2} EI_z \kappa^2 dx}_{\text{弯曲变形能（参考P48）}} - P\underbrace{\delta_v}_{\text{端部轴向位移（参考P61）}} + \cancel{\Pi_{\text{压缩}}}^{\,0}$$ 忽略（细长杆 inextension 假设）

$$= \int_0^l \frac{1}{2} EI_z v''^2 dx - P\int_0^l \frac{1}{2} v'^2 dx$$

我们先做标度（scaling）分析，再做具体的参数求解.

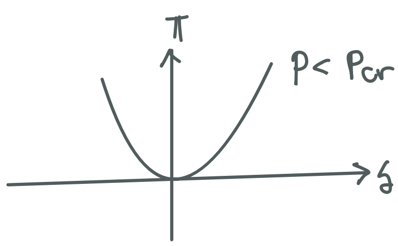


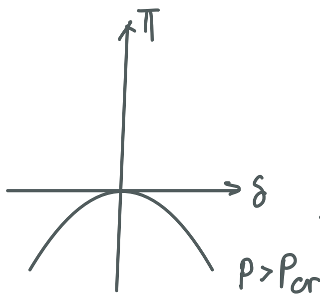
$$v \sim \delta, \quad v' \sim \delta/l, \quad v'' \sim \delta/l^2 \quad \rightarrow \quad \int_0^l \frac{1}{2} EI_z v''^2 dx \sim EI_z \frac{\delta^2}{l^3}, \quad \int_0^l v'^2 dx \sim \frac{\delta^2}{l}$$

$$\therefore \ \Pi \sim EI_z \frac{\delta^2}{l^3} - P\frac{\delta^2}{l} = \frac{1}{l}\left(\frac{EI_z}{l^2} \underbrace{-}_{\text{标度意义下的减号}} P\right)\delta^2$$

该简单分析可以告诉我们一系列有趣的力学行为，特别是：

- 存在一个临界 P_{cr}，其量级为 $\sim \frac{EI_z}{l^2}$
- 当 $P < P_{cr}$ 时，$\Pi = \alpha\delta^2$，$\alpha > 0$，$\frac{\partial \Pi}{\partial \delta} = 0 \rightarrow \delta = 0$，$\frac{\partial^2 \Pi}{\partial \delta^2} > 0$ → 存在弯曲变形为0的稳定解，即不会产生弯曲变形
- 当 $P > P_{cr}$ 时，$\Pi = \alpha\delta^2$，$\alpha < 0$，$\frac{\partial \Pi}{\partial \delta} = 0$ 但 $\frac{\partial^2 \Pi}{\partial \delta^2} < 0$，系统无稳定的弯曲变形解答.



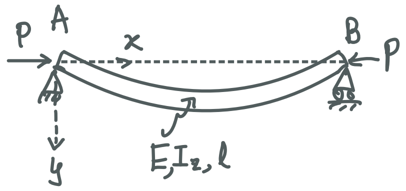


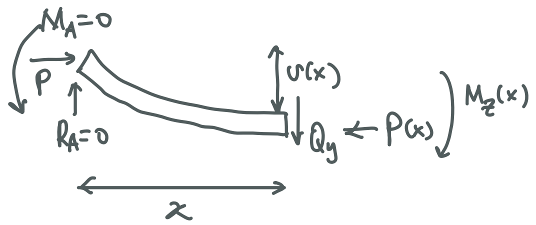
问题：如何确定 P_{cr}？

行动①：根据 $P < P_{cr}$ 时只存在 $\delta = 0$ 解

行动②：$P = P_{cr}$，$\frac{\partial \Pi}{\partial \delta} = \frac{\partial^2 \Pi}{\partial \delta^2} = 0$，$\delta$ 可为任意值

§6.2. 按特征值方法给出的压杆临界力



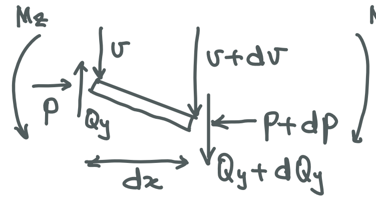


$P(x) \equiv P$

$Q_y = 0$

$\Rightarrow M_z + P v(x) = 0$
平衡方程

或采用



$\Rightarrow$

$\frac{dP}{dx}=0 \to P(x)=P(0)=P$

$\frac{dQ_y}{dx}=0 \to Q_y(x)=Q_y(0)=0$

$\frac{dM_z}{dx}+P\frac{dv}{dx}=0 \xrightarrow[\text{积分一次}]{} M_z + Pv = \cancel{C} \to 0$ $M_z(0)=0,\ v(0)=0$

存在轴力 P (压缩方向) 的挠曲轴方程： $EI_z v'' + Pv = 0$

subject to $v(0)=v(l)=0$

令 $k^2=\frac{P}{EI_z}$，ODE为 $v''+k^2v=0$，其通解为 $v = A\sin kx + B\cos kx$

$v(0)=0 \to B=0$

$v(l)=0 \to A\sin kl = 0 \to A=0$ 或 $k=\frac{n\pi}{l}$，i.e.，$P=n^2\pi^2\frac{EI_z}{l^2}$，$n=0,1,2\cdots$

若 $P\neq n^2\pi^2\frac{EI_z}{l^2}$，$A=0$，$v(x)\equiv 0$，只存在 $v=0$ 的稳定解.(平凡解)

若 $P=0$，通解为 $v=Cx+D \to v\equiv 0$

只有 $\underbrace{P=n^2\pi^2\frac{EI_z}{l^2}}_{\text{边值问题的特征值}}$，$n=1,2,3,\cdots$ 时，存在弯曲变形解答，且 A 即 $\delta=v(\frac{l}{2})$ 为任意值.

(我们想要寻找的 P_{cr})

$$\begin{bmatrix} 0 & 1 \\ \sin kl & \cos kl \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}=0$$

$A\neq 0, B\neq 0$ (非平凡解) 的条件为 $\begin{vmatrix} 0 & 1 \\ \sin kl & \cos kl \end{vmatrix}=0 \to \sin kl = 0$ 特征方程

最小的临界力或欧拉临界载荷 $\boxed{P_E=\frac{\pi^2 EI_z}{l^2}，\text{对应的形态为 } v=A\sin\frac{\pi x}{l}}$

A为任意值.

• $P=P_E$ 的物理含义（以纵-横弯曲为例）：

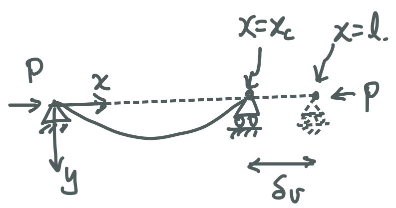
我们求解了该问题： $v(x)=e\left[\frac{1-\cos kl}{\sin kl}\sin kx-(1-\cos kx)\right]$

当 $P=P_E$ 时，$k=\frac{P}{EI_z}=\frac{\pi}{l}$ → $v(x)=\frac{2e\sin\pi x}{\sin\pi}-(1-\cos\frac{\pi x}{l})$ → $\delta=v(\frac{l}{2})=\frac{2e}{0}$

任意e（扰动）下，无论e多小，都给出无限大的δ （与实际不符－撑杆跳运动员正是利用杆在屈曲状态下的支撑力跃起 i.e. $k=\frac{P}{\delta}\neq0$）

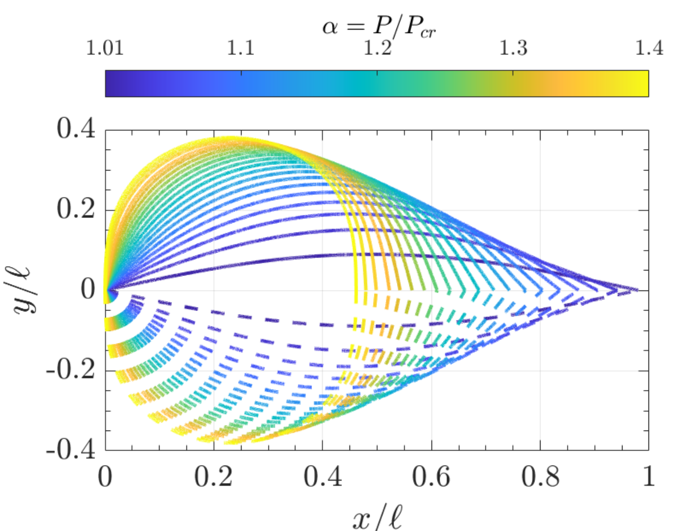
• $P\geq P_E$ 时的 $P-\delta$（压缩-挠度）曲线

失稳后，需采用大变形理论，并结合不可拉伸条件：压缩量 δ_v 全部转为弧长



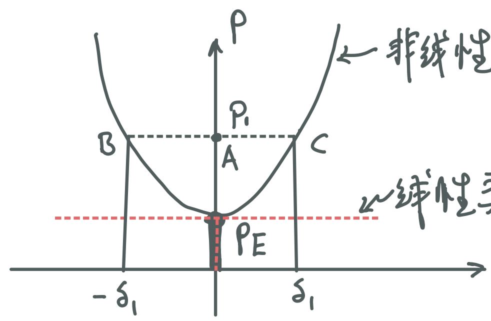
$$EI_z\frac{v''}{(1+v'^2)^{3/2}}+Pv=0$$

subject to
$$\begin{cases} v(0)=0 \\ v(x_c)=0 \quad \leftarrow \text{未知} \\ \int_0^{x_c}\sqrt{1+v'^2}\,dx=l \end{cases}$$



Shooting method √　BVP（病态、v=0解）

分岔点失稳（Bifurcation）　（参考教材P299）



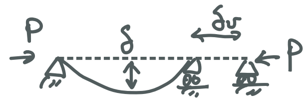
← 非线性理论 $K=\frac{v''}{(1+v'^2)^{3/2}}$

← 线性梁理论 $K=v''$ （δ为任意值）

势能？

当 $P>P_E$ 时，非线性理论可以给出两个"稳定"的平衡解答，为什么两个？为什么稳定？

我们仍然可以采用 Scaling 分析来定性的理解：



$$l = \int_0^{l-\delta_v}\sqrt{1+v'^2}\,dx$$

$$\simeq \int_0^{l-\delta_v}\left(1+\frac{1}{2}v'^2-\frac{1}{8}v'^4\right)dx$$

$$= l-\delta_v + \int_0^{l-\delta_v}\left(\frac{1}{2}v'^2-\frac{1}{8}v'^4\right)dx$$

$$\to \delta_v = \int_0^{\overset{\text{Correction}}{l-\delta_v}}\left(\frac{1}{2}v'^2-\underbrace{\frac{1}{8}v'^4}_{\text{H.O.T.}}\right)dx$$

$$K^2 = \left[\frac{v''}{(1+v'^2)^{3/2}}\right]^2 \simeq v''^2\left(1\overbrace{-3v'^2}^{\text{H.O.T.}}+\cdots\right)$$

$$\Rightarrow \Pi = U_{SE} - P\delta$$

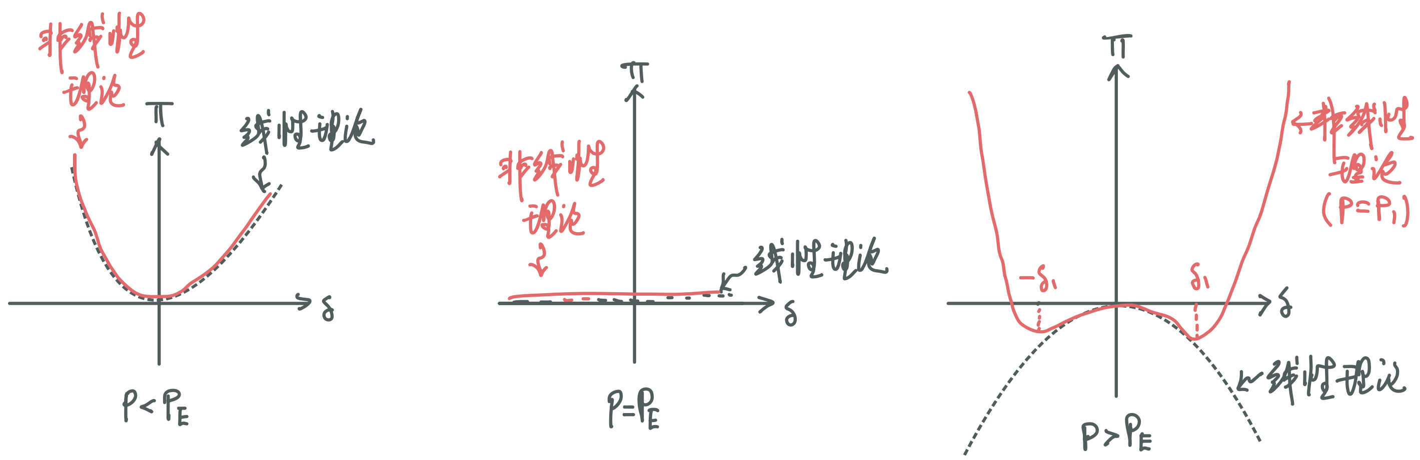
$$= \int_0^{l-\delta_v}\frac{1}{2}EI_zK^2\,dx - P\int_0^{l-\delta_v}\left(\frac{1}{2}v'^2-\frac{1}{8}v'^4\right)dx$$

$$= \int_0^{l-\delta_v}\left(\frac{1}{2}EI_z v''^2-\frac{1}{2}Pv'^2\right)+\underbrace{\left(-\frac{3}{2}EI_z v''^2v'^2+\frac{1}{8}Pv'^4\right)}_{\text{H.O.T.}}dx$$

考查 $\delta_v \ll l$，$\underbrace{v=\delta\sin\left(\pi\frac{x}{l}\right)}_{(\text{满足 } v(0)=v(l)=0)}$，不难看出高阶非线性项会给出 $\Pi\sim\delta^2+\delta^4$ 的形式。

具体的计算表明：$\Pi = \frac{1}{4l}\left(\frac{\pi^2EI_z}{l^2}-P\right)\delta^2+\frac{3\pi^4}{16l^3}\left(P-\frac{\pi^2EI_z}{l^2}\right)\delta^4$

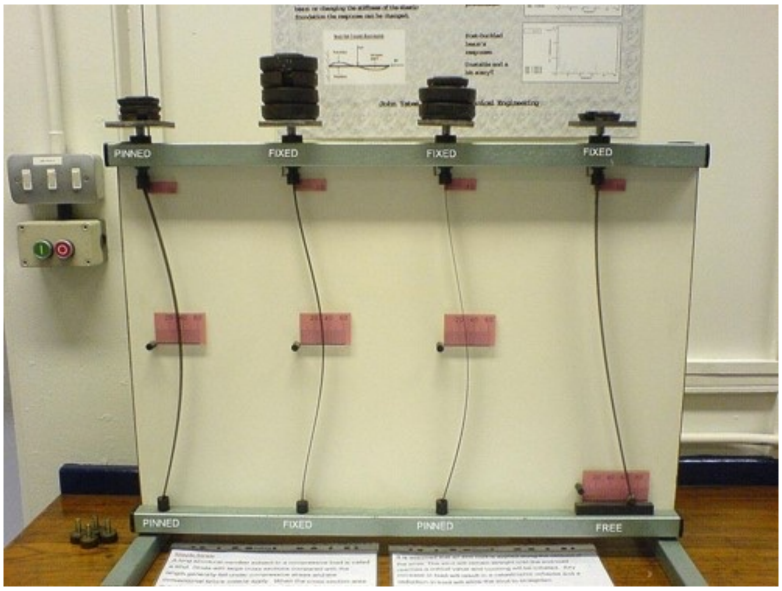
$$= \underbrace{\frac{1}{4l}(P_E-P)\delta^2}_{\text{线性小变形解答}}+\underbrace{\frac{3\pi^4}{16l^3}(P-P_E)\delta^4}_{\text{非线性大变形导致的高阶项}}$$



(注：此时 $\delta=0$ 为解，因此关注 $\frac{\delta}{l}\ll1$)

(注：此时 $\delta=0$ 为非稳定解，关注 $\frac{\delta}{l}\sim1$)

§6.3. 压杆在其它支承条件下的临界力



$\mu=1$ $\mu=0.5$ $\mu=0.7$ $\mu=2$

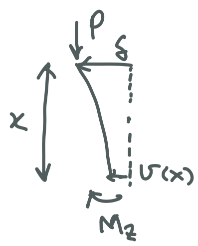
可采用特征值方法给出不同边界条件下的"第一"临界力。

$$P_E = \frac{\pi^2 EI_z}{(\mu l)^2}$$

μ: 长度折算系数

μl: 相当长度

• 自由-固支压杆



$$M_z = P(\delta - v)$$

$$\rightarrow EI_z v'' + Pv = P\delta \quad , \quad k^2 = \frac{P}{EI_z}$$

解答: $v = A\sin kx + B\cos kx + \delta$

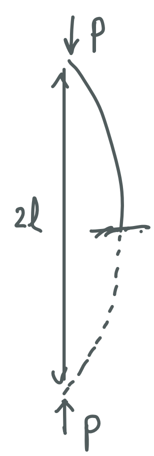
$v''(0) = 0 \rightarrow B = 0$

$v(l) = 0 \rightarrow A\sin kl + \delta = 0$

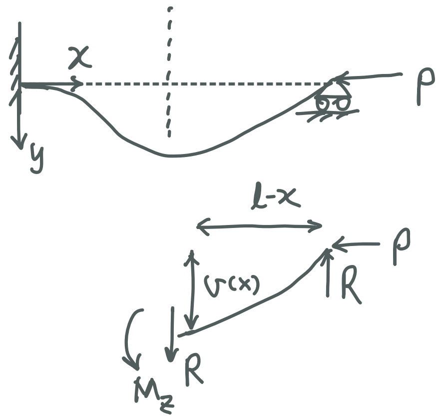
$v'(l) = 0 \rightarrow kA\cos kl = 0 \rightarrow A$ 或 $kl = \frac{(2n-1)\pi}{2}$, $n = 1, 2, \cdots$

第一个临界力: $kl = \frac{\pi}{2} \rightarrow P = \frac{\pi^2 EI_z}{(2l)^2}$, i.e., $\mu = 2$

对应的模态: $v(x) = \delta\left(1 - \sin\frac{\pi}{2}\frac{x}{l}\right)$



- 简支-固支压杆



$$EI_z v'' + Pv = -R(l-x), \quad k^2 = \frac{P}{EI_z}$$

$$\rightarrow \quad v(x) = A\sin kx + B\cos kx + \frac{R}{P}(x-l)$$

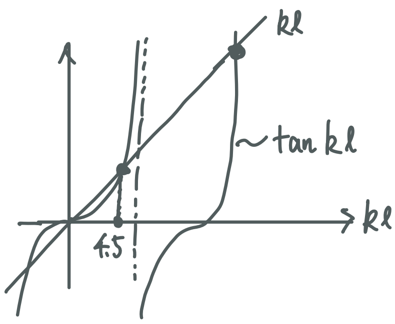
$$\left.\begin{aligned} v(0) = 0 &\rightarrow B = \frac{Rl}{P} \\ v'(0) = 0 &\rightarrow kA + \frac{R}{P} = 0 \end{aligned}\right\} \rightarrow kA + \frac{B}{l} = 0$$

$$v(l) = 0 \rightarrow A\sin kl + B\cos kl = 0$$

$$\begin{bmatrix} k & \frac{1}{l} \\ \sin kl & \cos kl \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

平凡解为 $A=B=R=0$

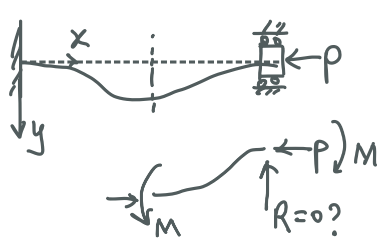
非平凡解存在，k需满足 $\underbrace{\tan kl = kl}_{\text{特征方程}}$



$$\rightarrow P_E = \frac{4.5^2 EI_z}{l^2} = \frac{\pi^2 EI_z}{(0.7l)^2}, \quad \mu = 0.7$$

对应模态：$v(x) = A\left[\sin\frac{4.5(l-x)}{l} + 0.976\frac{l-x}{l}\right]$

- 固支-固支压杆



$$EI_z v'' + Pv = M, \quad k^2 = \frac{P}{EI_z}$$

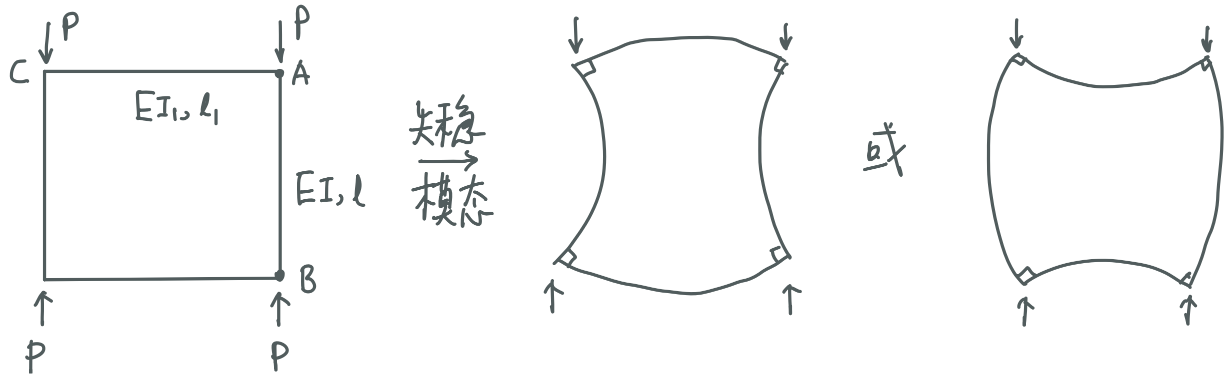
$$v = A\sin kx + B\cos kx + \frac{M}{P}$$

$$\begin{cases} v(0) = B + \frac{M}{P} = 0 \\ v'(0) = kA = 0 \\ v(l) = B\cos kl + \frac{M}{P} = 0 \rightarrow \cos kl = 1 \\ v'(l) = -kB\sin kl = 0 \rightarrow \sin kl = 0 \end{cases} \rightarrow kl = 2n\pi$$

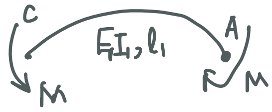
$$\rightarrow P_E = \frac{4\pi^2 EI_z}{l^2} = \frac{\pi^2 EI_z}{(0.5l)^2}, \quad \mu = 0.5$$

模态：$v(x) = B\left(\cos\frac{2\pi x}{l} - 1\right)$

• 框架失稳

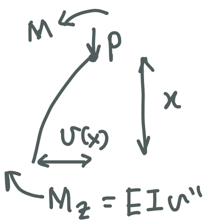


在A处，除了压力P作用外，还有AC梁弯曲导致的弯矩M



$$EIv''+Pv=M\ ,\quad k^2=\frac{P}{EI}$$

$$\to v(x)=A\sin kx+B\cos kx+\frac{M}{P}$$



$$\begin{cases} v(0)=B+\frac{M}{P}=0 \\ v'(\frac{l}{2})=kA\cos\frac{1}{2}kl-kB\sin\frac{1}{2}kl=0 \end{cases}$$

但是M未知 缺少条件？

考虑A处的转角： $U_{SE}=\dfrac{M^2 l_1}{2E_1I_1}=2\times\frac{1}{2}M\theta \to \theta=\dfrac{Ml_1}{2E_1I_1}$

转角连续： $v'(0)=kA=\dfrac{Ml_1}{2E_1I_1}=-\dfrac{BPl_1}{2E_1I_1}$

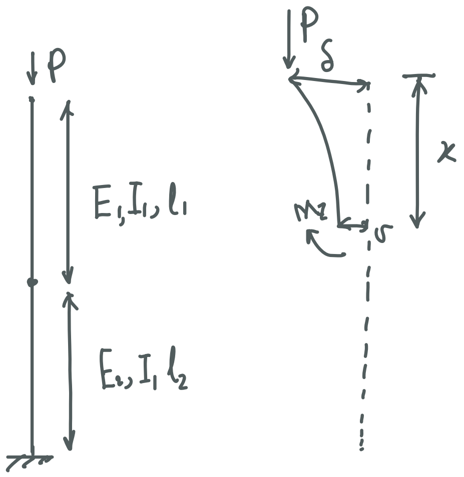
$$\begin{bmatrix} k & \frac{Pl_1}{2E_1I_1} \\ k\cos\frac{1}{2}kl & -k\sin\frac{1}{2}kl \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}=0 \to \tan\frac{1}{2}kl+\underbrace{\frac{Pl_1}{2kE_1I_1}}_{=\frac{1}{2}kl\cdot\frac{EIl_1}{E_1I_1l}}=0$$

Check：当 $\dfrac{EI}{l}\gg\dfrac{E_1I_1}{l_1}$ 时， $\frac{1}{2}kl\to\frac{\pi}{2}$ ， $P_E\to\dfrac{\pi^2EI}{l^2}$ （简支－简支压杆）

当 $\dfrac{EI}{l}\ll\dfrac{E_1I_1}{l_1}$ 时， $\frac{1}{2}kl\to\pi$ ， $P_E\to\dfrac{\pi^2EI}{(0.5l)^2}$ （固支－固支压杆）

当 $\dfrac{EI}{l}=\dfrac{E_1I_1}{l_1}$ 时， $\tan\frac{1}{2}kl+\frac{1}{2}kl\to0$ ， $P_E\to\dfrac{\pi^2EI}{(0.774l)^2}$

• 组合杆



$$\begin{cases} E_1I_1 v'' + Pv = P\delta, & 0<x<l_1 \\ E_2I_2 v'' + Pv = P\delta, & l_1<x<l_1+l_2 \end{cases}$$

$$v = \begin{cases} A\sin k_1 x + B\cos k_1 x + \delta, & 0<x<l_1 \\ C\sin k_2 x + D\cos k_2 x + \delta, & l_1<x<\underbrace{l_1+l_2}_{=l} \end{cases}$$

$v(0) = B + \delta = \delta$

$v(l) = C\sin k_2 l + D\cos k_2 l + \delta = 0$

$v'(l) = k_2 C\cos k_2 l - k_2 D\sin k_2 l = 0$

$v(l_1^-) = v(l_1^+)$　　　注意 $v''(l_1^-) = v''(l_1^+)$ 自动满足

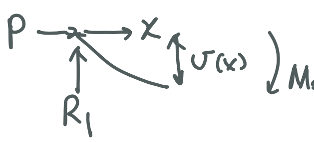
$v'(l_1^-) = v'(l_1^+)$

$$\begin{bmatrix} 1 & 0 & \sin k_2 l & \cos k_2 l \\ 0 & 0 & k_2\cos k_2 l & -k_2\sin k_2 l \\ 0 & \sin k_1 l_1 & -\sin k_2 l_1 & -\cos k_2 l_1 \\ 0 & k_1\cos k_1 l_1 & -k_2\cos k_2 l_1 & k_2\sin k_2 l_1 \end{bmatrix} \begin{bmatrix} \delta \\ A \\ C \\ D \end{bmatrix} = 0 \longrightarrow k_1\cos k_1 l_1 \cos k_1(2l_1+l_2) + k_2 \sin k_1 l_1 \sin k_2 l_2 = 0$$

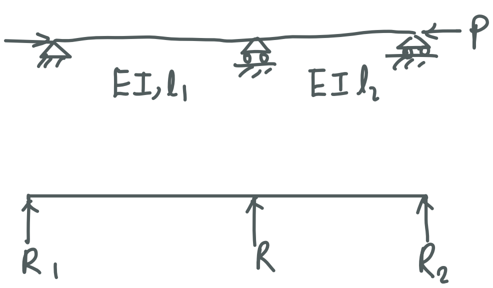
Check: $k_1 = k_2 = k$, $l_1 = l_2 = \frac{1}{2}l$

$\rightarrow \cos kl = 0$, $kl = \frac{\pi}{2}$ ✓

• 静不定杆



$\rightarrow EIv'' + Pv = -R_1 x$



$v(x) = A\sin kx + B\cos kx - \frac{R_1}{P}x$

$v(0) = 0 \rightarrow B = 0$

$v(l_1) = 0 \rightarrow A\sin kl_1 - \frac{R_1}{P}l_1 = 0$ ②

$R_1 + R_2 + R = 0$

$R_1 l_1 - R_2 l_2 = 0$ ①

同样，对于右侧部分，建立 $\overset{x}{\leftarrow}\downarrow_y$ 坐标系，可得

$v(x) = C\sin kx + D\cos kx - \frac{R_2}{P}x$

$v(0) = 0 \rightarrow D = 0$

$v(l_2) = 0 \rightarrow C\sin kl_2 - \frac{R_2}{P}l_2 = 0$ ③

未知数：R_1, R_2, R, A, B, C, D，但只有6个边条/方程

第7个方程来自于转角的连续性：$kA\cos kl_1 - \frac{R_1}{P} = -\left(kC\cos kl_2 - \frac{R_2}{P}\right)$ ⑦

负号因为x方向相反

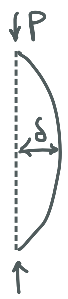
$$\begin{bmatrix} l_1 & -l_2 & 0 & 0 \\ -l_1/P & 0 & \sin kl_1 & 0 \\ 0 & -l_2/P & 0 & \sin kl_2 \\ -\frac{1}{P} & -\frac{1}{P} & k\cos kl_1 & k\cos kl_2 \end{bmatrix}\begin{bmatrix} R_1 \\ R_2 \\ A \\ C \end{bmatrix} = 0 \rightarrow kl_1\, kl_2 \sin k(l_1+l_2) = \sin kl_1 \sin kl_2 (kl_1 + kl_2)$$

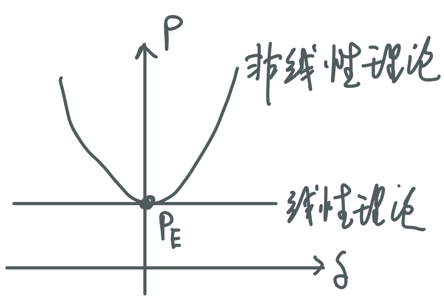
当 $l_1 = l_2 = l$ 时，$kl\sin 2kl + \cos 2kl - 1 = 0$ 或 $kl\sin kl\cos kl = \sin^2 kl$

- 第一模态 $kl = \pi$，$\mu = 1$
- 第二模态 $kl = 4.5$，$\mu = 0.7$

§6.4 分岔点失稳和极值点失稳

· 分岔点失稳





$P_E = \frac{\pi^2 EI_z}{(\mu l)^2}$ （欧拉临界力）

$\sigma_E = \frac{\pi^2 EI_z}{(\mu l)^2 A} = \frac{\pi^2 E}{\lambda^2}$ （临界应力）

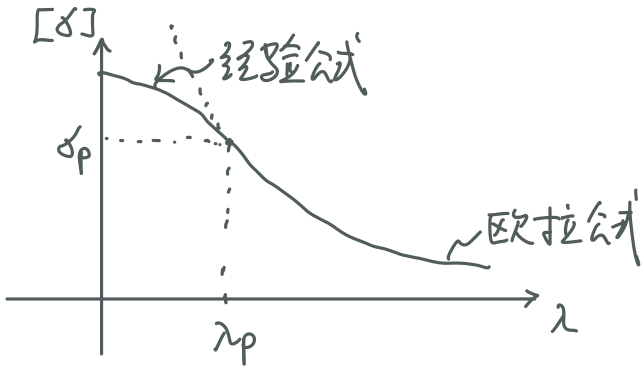
$\underbrace{\lambda = \frac{\mu l}{\sqrt{I_z/A}}}_{\text{柔度}} \sim \frac{l}{d}$ （长细比）

欧拉公式适用需 $\sigma_E < \sigma_p$（屈服应力）或 $\lambda > \lambda_p$，$\lambda_p = \left(\frac{\pi^2 E}{\sigma_p}\right)^{1/2}$

- $\lambda > \lambda_p$，$[\sigma] < \sigma_E$（否则产生欧拉失稳）

- $\lambda < \lambda_p$ $\quad [\sigma] < \sigma_r = \sigma_s\left[1-\alpha\left(\frac{\lambda}{\lambda_c}\right)^2\right]$（否则产生塑性+失稳）

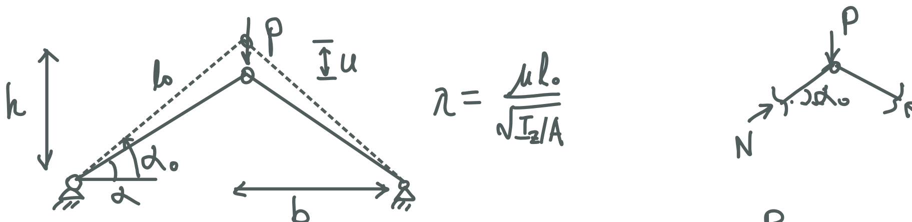
$\alpha \sim 0.43$，$\lambda_c \sim \sqrt{\frac{\pi^2 E}{0.57\sigma_s}}$（经验公式）



· 极值点失稳：1930年代，Von Karman & 钱学森讨论了浅壳的另外一种失稳行为

Tisen & Karman (1939), Karman & Tsien (1941)

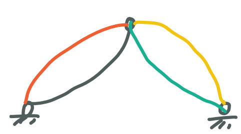
我们通过浅桁架（铰支）来定量的理解：



$\lambda = \frac{\mu l_0}{\sqrt{I_z/A}}$

"浅"代表 $|\alpha| \ll 1$，$\sin\alpha \sim \alpha$

$N = \frac{P}{2\sin\alpha_0} \simeq \frac{P}{2\alpha_0}$

→ 当 $N = N_{cr} = EA\frac{\pi^2}{\lambda}$ 时，即 $P_{cr}^E = 2\alpha_0 EA\frac{\pi^2}{\lambda}$，发生  （4种模态）

考虑在P的作用下，也会产生位移 u，使 $\alpha_0 \to \alpha$，$l_0 \to l$，P-u关系？能量法？

• 过去，我们认为小变形，$\alpha_0 \sim \alpha$，$l_0 \sim l$，得到线性力学响应：

$$\frac{1}{2}P\cdot u = 2\times\frac{1}{2}\frac{N^2 l_0}{EA} = \frac{P^2 l_0}{4\alpha_0^2 EA} \to P = \frac{2\alpha_0^2 EA}{l_0}u = \frac{2EAh^2}{l_0^3}u$$

• 当 $\alpha_0 \ll 1$ 时，很容易导致 $\Delta\alpha \sim \frac{u}{l_0} \sim \alpha_0$，则需考虑有限变形，此时：

$$\left.\begin{aligned} l_0 &= \frac{b}{\cos\alpha_0} \simeq b\left(1+\frac{1}{2}\alpha_0^2+\cdots\right) \\ l &= \frac{b}{\cos\alpha} \simeq b\left(1+\frac{1}{2}\alpha^2+\cdots\right) \end{aligned}\right\} \to \underbrace{\Delta l}_{\text{缩短量}} \simeq \frac{b}{2}(\alpha_0^2-\alpha^2)$$

$$\varepsilon = \frac{\Delta l}{l_0} = \frac{b(\alpha_0^2-\alpha^2)}{2b(1+\frac{1}{2}\alpha_0^2)} = \frac{1}{2}(\alpha_0^2-\alpha^2) + O(\alpha_0^4)$$

几何关系还能告诉我们 $\alpha = \frac{h-u}{l_0} = \frac{h}{l_0}\left(1-\frac{u}{h}\right) = \alpha_0\left(1-\frac{u}{h}\right)$

平衡关系给出 $P = 2N\alpha = 2EA\varepsilon\alpha = EA(\alpha_0^2-\alpha^2)\cdot\alpha$

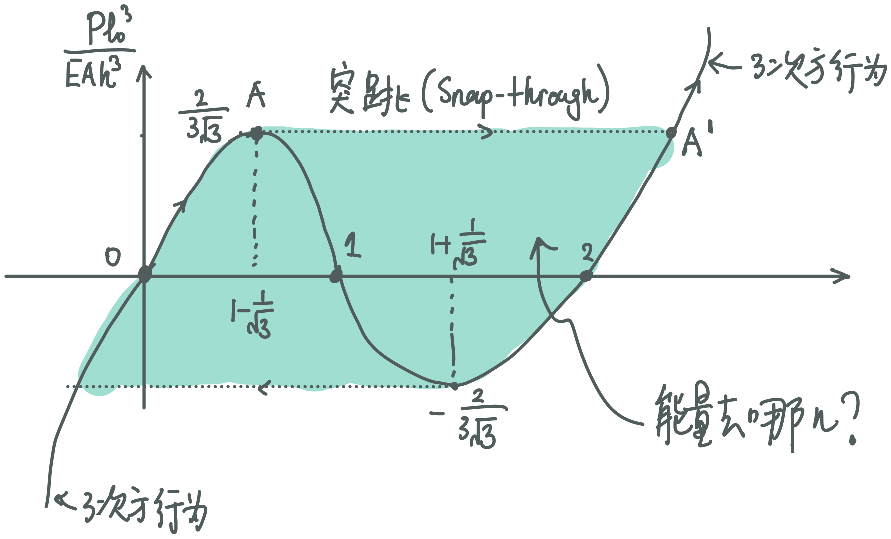
$$\to P = EA\alpha_0^3\left(1-\left(1-\frac{u}{h}\right)^2\right)\cdot\left(1-\frac{u}{h}\right)$$

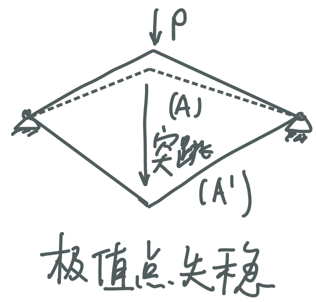
$$= \frac{EAh^3}{l_0^3}\left(2\frac{u}{h}-\frac{u^2}{h^2}\right)\left(1-\frac{u}{h}\right)$$

$$= \frac{EA}{l_0^3}u(u-h)(u-2h)$$ ← 我们只假设了 $|\alpha_0|, |\alpha| \ll 1$，$\sin\alpha \sim \alpha$，$\cos\alpha \sim 1-\frac{1}{2}\alpha$

$$= \frac{EA}{l_0^3}(u^3-3hu^2+2h^2u)$$

$$\to P = \begin{cases} \dfrac{2EAh^2}{l_0^2}u, & \text{当 } \dfrac{u}{h} \ll 1 \text{（线性）} ✓ \\ \dfrac{EA}{l_0^3}u^3, & \text{当 } \dfrac{u}{h} \gg 1 \text{（非线性）} \end{cases}$$





$$\rightarrow P_{cr}^{k} = \frac{2}{3\sqrt{3}}\frac{EAh^3}{l_0^2}, \quad P_{cr}^{E} = 2\alpha_0 EA\frac{\pi^2}{\lambda}$$

$$\rightarrow P_{cr}^{k}/P_{cr}^{E} = \frac{1}{3\sqrt{3}\pi^2}\alpha_0^2\lambda^2 = \frac{\mu^2}{3\sqrt{3}\pi^2}\frac{h^2}{I_z/A} \xlongequal[I_z=\frac{\pi}{4}R^4]{\mu=1} \approx 13\cdot\left(\frac{h}{R}\right)^2$$

结构"细长比"　　材料"长细比"

- $h \gg R$, $P_{cr}^{k} \gg P_{cr}^{E}$, 分岔点失稳
- $h \ll R$, $P_{cr}^{k} \ll P_{cr}^{E}$, 极值点失稳