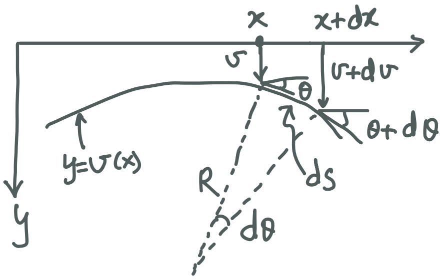
# 第五章 弯曲变形

## §5.1. 挠曲轴的微分方程

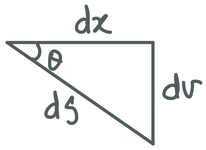
在上一章，我们定义了挠度 $v$、曲率 $\kappa=\frac{1}{R}$ (正方向与 $M$ 一致)，得到 $M=EI\kappa$.

注意教材上 $\kappa$、$M$ 方向不一致，所以 $M=-EI\kappa$




$\theta$：挠曲轴切线与 $x$ 轴夹角（转角）

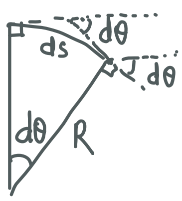
ds, dv, dx 关系?




$\rightarrow \tan\theta = \frac{dv}{dx} = v'$ or $\theta = \arctan v'(x)$

$\rightarrow ds = \sqrt{dx^2+dv^2} = dx\sqrt{1+v'^2}$

为什么 ds 扫过 dθ? 考虑 $\theta=0$




$\rightarrow R\,d\theta = ds$

$$\therefore \frac{1}{R} = \frac{d\theta}{ds} = \frac{d\theta}{dx}\cdot\frac{dx}{ds}$$

$$= \frac{v''}{1+v'^2}\cdot\frac{1}{\sqrt{1+v'^2}}$$

$$= \frac{v''}{(1+v'^2)^{3/2}}$$

因此几何关系可以给出 $M_z(x) = EI_z\frac{v''}{(1+v'^2)^{3/2}}$

考虑到 $|v'| \sim |\frac{v}{l}| \ll 1$，$\kappa \simeq v'' \rightarrow$ $\boxed{EI_z v'' = M_z(x),\quad EI_z v''' = -Q_y(x),\quad EI_z v'''' = q(x)}$ ☆

Euler-Bernoulli Beam theory

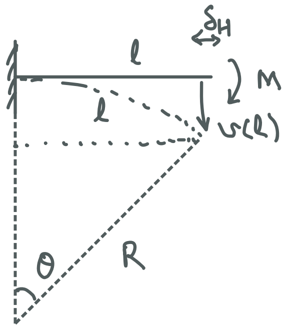
在解答这个4阶ODE前，我们先思考以下几个性质/问题：

• 叠加原理 (superposition)

$$EI_z v_{①}'''' = q_{①}(x)$$
$$EI_z v_{②}'''' = q_{②}(x)$$
$$\longrightarrow EI_z\left(v_{①}'''' + v_{②}''''\right) = q_{①}(x) + q_{②}(x)$$

可采用叠加法解答问题

• 线性化后的精度？考查如下特例




$\theta = l/R$

$M_z(x) \equiv M$

线性化的方程 $EI_z v'' = M$

$$EI_z v' = Mx + \cancel{C_0} \xrightarrow{} 0 \quad v'(0) = 0$$

$$EI_z v = \frac{1}{2}Mx^2 + \cancel{C_1} \xrightarrow{} 0 \quad v(0) = 0$$

$$\longrightarrow v(l) = \frac{M}{2EI_z} l^2$$

非线性方程 $\frac{1}{R} = \frac{M}{EI_z} \to R \equiv \frac{EI_z}{M}$

$$v_1(l) = R - R\cos\theta = \frac{EI_z}{M}\left(1 - \cos\frac{Ml}{EI_z}\right)$$

$$= \frac{EI_z}{M}\left[1 - \left(1 - \frac{1}{2}\left(\frac{Ml}{EI_z}\right)^2 + \frac{1}{24}\left(\frac{Ml}{EI_z}\right)^4 + O(\varepsilon^6)\right)\right]$$

$$= \frac{Ml^2}{2EI_z} - \frac{1}{24}\frac{M^3 l^4}{(EI_z)^3}$$

相对误差

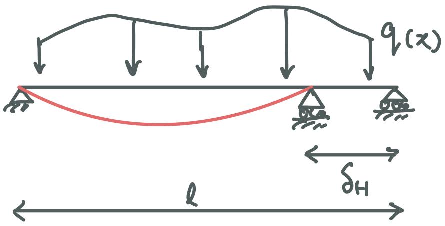
$$R.E. = \frac{v - v_1}{v} = \frac{1}{12}\frac{M^2 l^2}{(EI_z)^2} = \frac{1}{3}\left[\frac{v(l)}{l}\right]^2$$

注意 $\delta_H = l - R\sin\theta = l - R\left(\frac{l}{R} - \frac{1}{6}\frac{l^3}{R^3} + \cdots\right)$

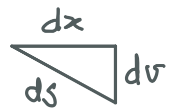
$$\simeq \frac{1}{6}\frac{l^3}{R^2} = \frac{1}{6}\frac{M^2 l^3}{(EI_z)^2} = \frac{2}{3}\frac{v^2(l)}{l} \sim O(v'^2 \times l)$$ why?

| $v(l)/l$ | R.E. | $\delta_H/l$ |
|---|---|---|
| 1% | $\sim 10^{-4}$ | $\sim 10^{-4}$ |
| 10% | $\sim 10^{-2}$ | $\sim 10^{-2}$ |
| 30% | 0.03 | 0.06 |
| 50% | 0.08 | 0.16 |
| 1 | 0.33 | 0.66 |

- 为什么水平位移 $\sim O(v'^2 \times l)$?

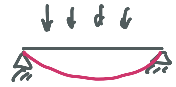



$$ds = \sqrt{dx^2 + dv^2} \underset{\text{Moderate rotation}}{\overset{\theta^2 = v'^2 \ll 1}{=\!=}} dx\left(1 + \frac{1}{2}v'^2\right)$$




$$\int_0^l ds = \int_0^l \left(1 + \frac{1}{2}v'^2\right)dx = l + \int_0^l \frac{1}{2}v'^2 dx > l$$

中性轴被拉长? 



→存在轴力

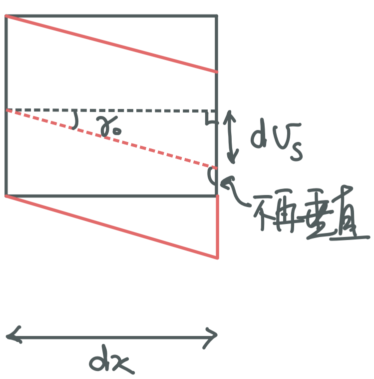
滑动铰支 → 轴力=0

$$l = \int_0^{l-\delta_H} ds = l - \delta_H + \int_0^{l-\delta_H} \frac{1}{2}v'^2 dx \rightarrow \boxed{\delta_H = \int_0^l \frac{1}{2}v'^2 dx}$$

中性轴长度不变 $(\delta_H \ll l)$

- 剪力对挠度有多少影响?

切应变使得平截面假设不再成立，取中性轴附近的微元




剪力引起的附加挠度

$$\frac{dv_s}{dx} = \gamma_0 = \frac{\tau_{xy}(y=0)}{G} = \alpha \frac{Q_y}{GA}, \quad \alpha = \begin{cases} \frac{3}{2} & \text{矩形截面} \\ \frac{4}{3} & \text{圆形截面} \end{cases}$$

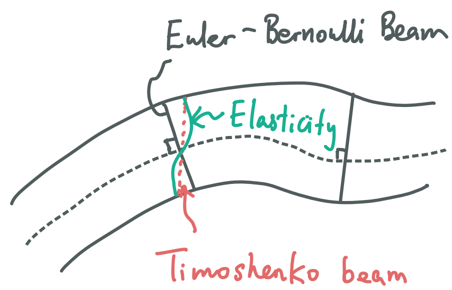
$$\frac{d^2 v_s}{dx^2} = -\alpha \frac{q}{GA}$$

$$v = \underbrace{v_b}_{\text{弯曲}} + \underbrace{v_s}_{\text{剪切}}$$

$$\rightarrow v'' = v_b'' + v_s''$$

$$= \frac{M_z}{EI_z} - \frac{\alpha q}{GA}$$

$$= \frac{M_z}{EI_z}\left(1 - \frac{\alpha q E I_z}{M_z G A}\right)$$



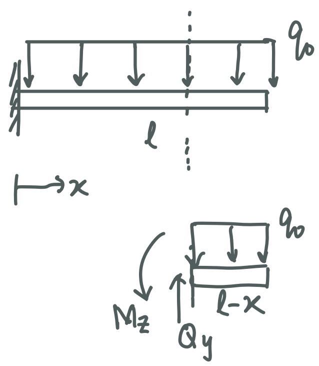

修正项: $\frac{\alpha q E I_z}{M_z G A} \sim \frac{1 \cdot q \cdot E \cdot h^3 b}{q l^2 \cdot E \cdot hb} = O\left(\frac{h^2}{l^2}\right)$

仅对短梁比较重要!

## §5.2. 弯曲方程的积分

我们已经推导了平衡方程：$EI_z v'' = M_z$，$EI_z v''' = -Q_y$，$EI_z v'''' = q$，可采用任何一个进行积分求解。

### 悬臂梁




$$M_z(x) = \frac{1}{2} q_0 (l-x)^2$$

$$v'' = \frac{q_0}{2EI_z}(l-x)^2$$

$$v' = \frac{-q_0}{6EI_z}(l-x)^3 + C_1, \quad v'(0)=0 \rightarrow C_1 = \frac{q_0 l^3}{6EI_z}$$

$$v = \frac{+q_0}{24EI_z}(l-x)^4 + \frac{q_0 l^3}{6EI_z}x + C_2, \quad v(0)=0 \rightarrow C_2 = -\frac{q_0 l^4}{24EI_z}$$

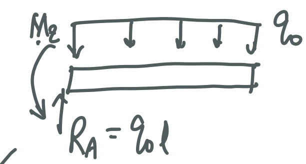
$$\rightarrow v(x) = \frac{q_0}{24EI_z}\left[-(l-x)^4 + 4l^3 x - l^4\right]$$

$$v_{max} = v(l) = \frac{q_0 l^4}{8EI_z}$$

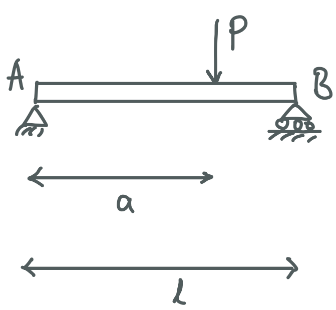
$$\theta_{max} = v'(l) = \frac{q_0 l^3}{6EI_z}$$

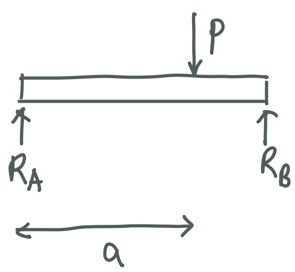
$$Q_y(x=0) = -EI_z v'''(0) = q_0 l$$

$$= R_A \ \checkmark$$




### 简支梁




平衡：

$$R_A = \frac{P(l-a)}{l}, \quad R_B = \frac{Pa}{l}.$$

$0 \le x < a \qquad q(x) = 0$

$$\frac{dQ_y}{dx} = -q \rightarrow Q_y(x) = C_1$$

$$\frac{dM_z}{dx} = -Q_y \rightarrow M_z(x) = -C_1 x + C_2$$

$$EI_z \frac{d\theta}{dx} = M_z \rightarrow EI\theta(x) = -\frac{1}{2}C_1 x^2 + C_2 x + C_3$$

$$\frac{dv}{dx} = \theta \rightarrow EIv(x) = -\frac{1}{6}C_1 x^3 + \frac{1}{2}C_2 x^2 + C_3 x + C_4$$

边界条件：$Q_y(0) = \frac{P(l-a)}{l} \rightarrow C_1 = \frac{P(l-a)}{l}$

$M_z(0) = 0 \rightarrow C_2 = 0$ $\qquad$ $\theta(0)$未知！

$v(0) = 0 \rightarrow C_4 = 0$

$a < x \le l \qquad q(x) = 0$

$$\rightarrow Q_y(x) = d_1$$

$$M_z(x) = -d_1 x + d_2$$

$$EI_z\theta(x) = -\frac{1}{2}d_1 x^2 + d_2 x + d_3$$

$$EI_z v(x) = -\frac{1}{6}d_1 x^3 + \frac{1}{2}d_2 x^2 + d_3 x + d_4$$

边界条件：$Q_y(l) = -\frac{Pa}{l} \rightarrow d_1 = -\frac{Pa}{l}$

$M_z(l) = 0 \rightarrow d_2 = -Pa$ $\qquad$ $\theta(l)$未知！

$v(l) = 0 \rightarrow \frac{1}{6}Pal^2 - \frac{1}{2}Pal^2 + d_3 l + d_4 = 0$

目前仍缺少2个条件来确定所有的积分常数 $C_1, C_2, C_3, C_4, d_1, d_2, d_3, d_4$。我们在$Q_y$, $M_z$图部分讲过跳跃(不连续条件)：

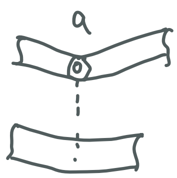
$$Q_y(a^+) - Q_y(a^-) = -P \quad \xleftarrow{\text{自动满足}} \quad Q_y(a^+) = -\frac{Pa}{l},\ Q_y(a^-) = \frac{P(l-a)}{l}$$

$$M_z(a^+) - M_z(a^-) = -M_0 = 0 \quad \xleftarrow{\text{自动满足}} \quad M_z(a^+) = \frac{Pa^2}{l} - Pa,\ M_z(a^-) = -\frac{P(l-a)a}{l}$$

$$= \frac{Pa}{l}(a-l)$$

真正未使用过的条件来自于 $v$，$\theta$ 的匹配条件 (Matching conditions)

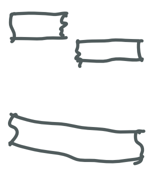
$\theta$: 




铰链无法传递力矩 → $M_z(a^+) = M_z(a^-) = 0$

无铰链，转角连续 → $\theta(a^+) = \theta(a^-)$

$v$: 



有断口无剪力 → $Q_y(a^+) = Q_y(a^-) = 0$

无断口位移连续 → $v(a^+) = v(a^-)$

对于我们的问题，$v$ 和 $\theta$ 都连续：

$$\theta:\ -\frac{1}{2}\frac{P(l-a)}{l}a^2 + C_3 = \frac{1}{2}\frac{Pa}{l}a^2 - Pa^2 + d_3 \rightarrow d_3 - C_3 = \frac{1}{2}Pa^2$$

$$v:\ -\frac{1}{6}\frac{P(l-a)}{l}a^3 + C_3 a = \frac{1}{6}\frac{Pa^4}{l} - \frac{1}{2}Pa^3 + d_3 a + d_4 \rightarrow a(d_3 - C_3) + d_4 = \frac{1}{3}Pa^3$$

$$\rightarrow d_4 = \frac{-1}{6}Pa^3,\ \text{代入 } v(l) = 0 \rightarrow d_3 = \frac{1}{3}Pal + \frac{1}{6}\frac{Pa^3}{l}$$

$$\rightarrow C_3 = \frac{-1}{2}Pa^2 + \frac{1}{3}Pal + \frac{1}{6}\frac{Pa^3}{l}$$

$$\rightarrow EIv(x) = \begin{cases} -\frac{1}{6}\frac{P(l-a)}{l}x^3 + P\left(-\frac{1}{2}a^2 + \frac{1}{3}al + \frac{1}{6}\frac{a^3}{l}\right)x, & 0 \le x < a \\ \frac{1}{6}\frac{Pa}{l}x^3 - \frac{1}{2}Pax^2 + P\left(\frac{1}{3}al + \frac{1}{6}\frac{a^3}{l}\right)x - \frac{1}{6}Pa^3, & a < x \le l \end{cases}$$

这一方法可以清晰的展示概念，但步骤过于繁琐，是否有简化的方法？—特殊函数

$$q(x) = P\varphi_{-1}(x-a)$$

$$Q_y(x) = -P\varphi_0(x-a) + \underbrace{Q_y(0)}_{\frac{P(l-a)}{l}}$$

$$M_z(x) = +P\varphi_1(x-a) - \frac{P(l-a)}{l}x + \cancel{M_z(0)} \to 0$$

$$EI_z\theta = P\varphi_2(x-a) - \frac{P(l-a)}{2l}x^2 + e_1$$

$$EI_z v = P\varphi_3(x-a) - \frac{P(l-a)}{6l}x^3 + e_1 x + \cancel{e_2} \to 0 = v(0)$$

$$EI_z v(x=l) = \frac{1}{6}P(l-a)^3 - \frac{1}{6}P(l-a)l^2 + e_1 l = 0 \to e_1 = P\left(-\frac{1}{2}a^2 + \frac{1}{3}la + \frac{1}{6}\frac{a^3}{l}\right) = C_3 \checkmark$$

Check

$$EI_z v(x>a) = \frac{1}{6}P(x-a)^3 - \frac{P(l-a)}{6l}x^3 + e_1 x$$

$$= \frac{1}{6}\frac{Pa}{l}x^3 - \frac{1}{2}Pax^2 + P\left(\frac{1}{3}la + \frac{1}{6}\frac{a^3}{l}\right)x - \frac{1}{6}Pa^3 \quad \checkmark \text{ Check.}$$

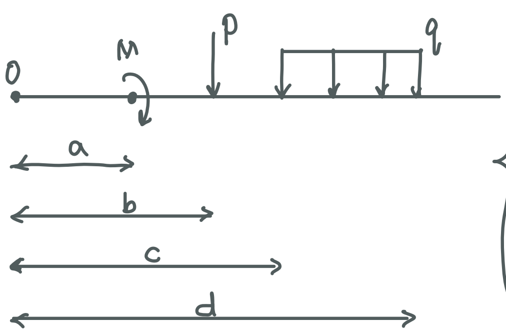
注意：该问题的最大挠度发生在 $x = \sqrt{\frac{2la-a^2}{3}}$ 处（而不是 $x=a$ 处），

$$v_{max} = \frac{P(l-a)(2la-a^2)^{3/2}}{9\sqrt{3}\,l\,EI_z}$$

（能量法只能给出a点的位移，而不是 $v_{max}$）

$$\boxed{\text{当 } a=\tfrac{1}{2}l \text{ 时，} v_{max} = \frac{1}{48}\frac{Pl^3}{EI_z}, \quad \theta_{max} = \theta(0) = -\theta(l) = \frac{Pl^2}{16EI_z}}$$

## 更为一般的情况

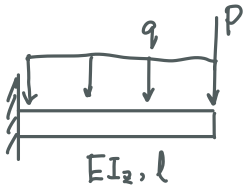



$$\begin{cases} q(x) = -M\varphi_{-2}(x-a) + P\varphi_{-1}(x-b) + q\varphi_0(x-c) - q\varphi_0(x-d) \\ Q_y(x) = M\varphi_{-1}(x-a) - P\varphi_0(x-b) - q\varphi_1(x-c) + q\varphi_1(x-d) + 0 \\ M_z(x) = -M\varphi_0(x-a) + P\varphi_1(x-b) + q\varphi_2(x-c) - q\varphi_2(x-d) + 0 \end{cases}$$

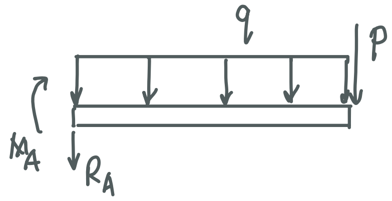
$$\rightarrow v'(x) = v'(0) + \frac{1}{EI_z}\left[-M\varphi_1(x-a) + P\varphi_2(x-b) + q\varphi_3(x-c) - q\varphi_3(x-d)\right]$$

$$\boxed{v(x) = v(0) + v'(0)x + \frac{1}{EI_z}\left[-M\varphi_2(x-a) + P\varphi_3(x-b) + q\varphi_4(x-c) - q\varphi_4(x-d)\right]}$$

例 1




受力分析




$R_A = -P - ql$

$M_A = -Pl - \frac{1}{2}ql^2$

此处特意将 $M_A, R_A$ 方向标注为与上述公式相同的方向

套用公式： $EI_z v(x) = EI_z \cancelto{0}{v(0)} + EI_z \cancelto{0}{v'(0)}\, x$

$$- M_A\varphi_2(x-0) + R_A\varphi_3(x-0) + \cancelto{0}{P\varphi_3(x-l)} + q\varphi_4(x-0) - \cancelto{0}{q\varphi_4(x-l)}$$

$$EI_z v(x) = \left(Pl + \frac{1}{2}ql^2\right)\frac{1}{2}x^2 - (P+ql)\frac{1}{3!}x^3 + q\frac{1}{4!}x^4$$

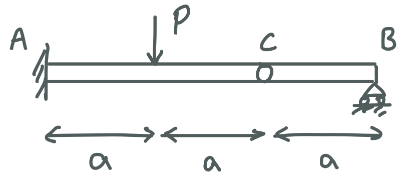
$$= \underbrace{\left(-\frac{1}{6}Px^3 + \frac{1}{2}Plx^2\right)}_{P\neq 0,\ q=0\text{时的解答}} + \underbrace{\left(-\frac{1}{6}qlx^3 + \frac{1}{4}ql^2x^2 + \frac{1}{24}qx^4\right)}_{P=0,\ q\neq 0\text{时的解答}}$$

$$v_{max} = v(x=l) = \underbrace{\frac{Pl^3}{3EI_z}}_{\text{期中考试1}} + \underbrace{\frac{ql^4}{8EI_z}}_{\text{上节课内容}} \quad (\text{叠加原理})$$

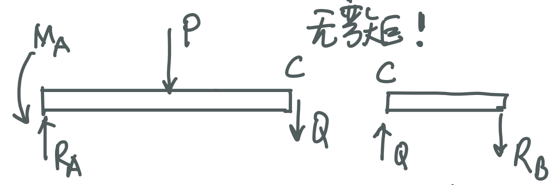
由P引起的弯矩

能量法求 $v_P$： $\frac{1}{2}P\cdot v_P = \int_0^l \frac{1}{2}\frac{M_P^2}{EI_z}dx = \int_0^l \frac{1}{2}\frac{(Px)^2}{EI_z}dx = \frac{P^2l^3}{6EI_z} \rightarrow v_P = \frac{Pl^3}{3EI_z}$

例2




FBD
受力分析




无弯矩!

$\Sigma F_y = 0 \rightarrow R_A = P$

$\Sigma M_z^A = 0 \rightarrow M_A = Pa$

$\Sigma F_y = 0 \rightarrow Q = R_B$

$\Sigma M_z^C = 0 \rightarrow R_B = Q = 0$

$0 \le x < 2a$:

$$q(x) = P\varphi_{-1}(x-a)$$

$$Q_y(x) = -P\varphi_0(x-a) + \overset{=R_A=P}{\cancel{Q_y(0)}}$$

$$M_z(x) = \overset{=M_A=Pa}{\cancel{M_z(0)}} - Px + P\varphi_1(x-a)$$

$$EI_z v'(x) = EI_z \overset{0}{\cancel{v'(0)}} + Pax - \frac{1}{2}Px^2 + P\varphi_2(x-a)$$

$$EI_z v(x) = EI_z \overset{0}{\cancel{v(0)}} + \frac{1}{2}Pax^2 - \frac{1}{6}Px^3 + P\varphi_3(x-a)$$

$2a < x \le 3a$:

$$q(x) = 0$$

$$Q_y(x) = \overset{=Q=0}{\cancel{Q_y(2a)}}$$

$$M_z(x) = \overset{0}{\cancel{M_z(2a)}}$$

$$v'(x) = C_1$$ ← 积分常数，待定 ($\theta$)

$$v(x) = C_1 x + C_2$$ (刚性转动)

边界/匹配条件:

$$v(x=3a) = 0 \rightarrow C_2 = -3aC_1$$

$$\underbrace{v(x=2a^-) = v(x=2a^+)}_{\text{未断开}} \rightarrow \underbrace{2aC_1 + C_2}_{-aC_1} = \frac{1}{EI_z}\left(\frac{1}{2}Pa\cdot 4a^2 - \frac{1}{6}P\cdot 8a^3 + \frac{1}{6}P\cdot a^3\right) = \frac{5}{6EI_z}Pa^3$$

$\Uparrow$

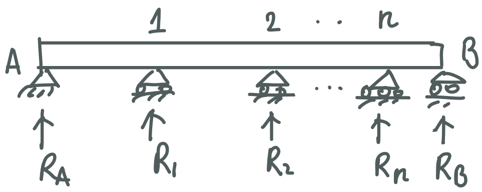
$$C_1 = -\frac{5}{6EI_z}Pa^2$$

$$C_2 = \frac{5}{2EI_z}Pa^3$$

$\uparrow$

$$EI_z v(x>2a) = -\frac{5}{6}Pa^2x + \frac{5}{2}Pa^3$$

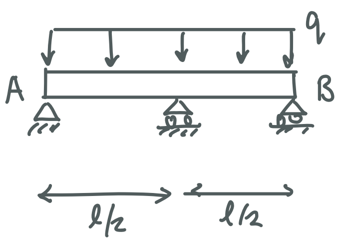
# §5.3、简单的静不定问题



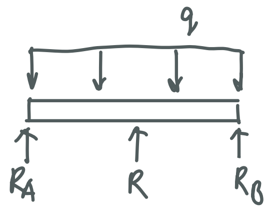

可将"多余"的n个支反力代入挠度公式

然后采用 $v_1 = v_2 = \cdots = v_n = 0$ 求解.

例3




FBD:




$$\Rightarrow \begin{cases} R_A + R_B + R = ql \\ R_A = R_B = \dfrac{ql-R}{2} \quad (\text{对称}) \end{cases}$$

$$EI_z v(x) = \cancel{EI_z v(0)}^{\,0} + EI_z \underbrace{v'(0)}_{\text{未知}} x$$

$$- \frac{1}{2}(ql-R)\varphi_3(x-0) - R\varphi_3\left(x-\frac{l}{2}\right) - \frac{1}{2}(ql-R)\cancel{\varphi_3(x-l)}^{\,0 \text{ since } x \le l} + q\varphi_4(x-0)$$

边界条件:

$$EI_z v\left(\frac{l}{2}\right) = EI_z v'(0)\frac{l}{2} - \frac{1}{2}(ql-R)\frac{1}{3!}\left(\frac{l}{2}\right)^3 + q\frac{1}{4!}\left(\frac{l}{2}\right)^4 = 0$$

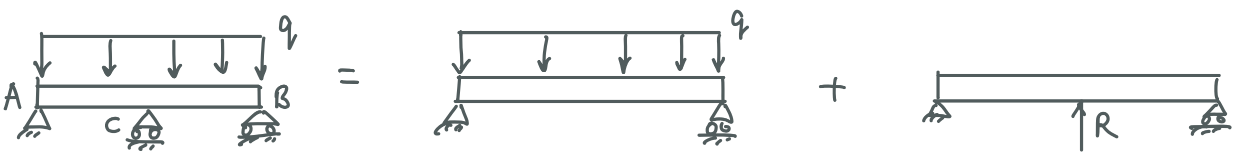
$$EI_z v(l) = EI_z v'(0)\cdot l - \frac{1}{2}(ql-R)\frac{1}{3!}l^3 - R\frac{1}{3!}\left(\frac{l}{2}\right)^3 + \frac{q}{4!}l^4 = 0$$

$$\rightarrow \begin{cases} v'(0) = \dfrac{ql^3}{384EI_z} \\ R = \dfrac{5}{8}ql \end{cases}$$

$$\Downarrow$$

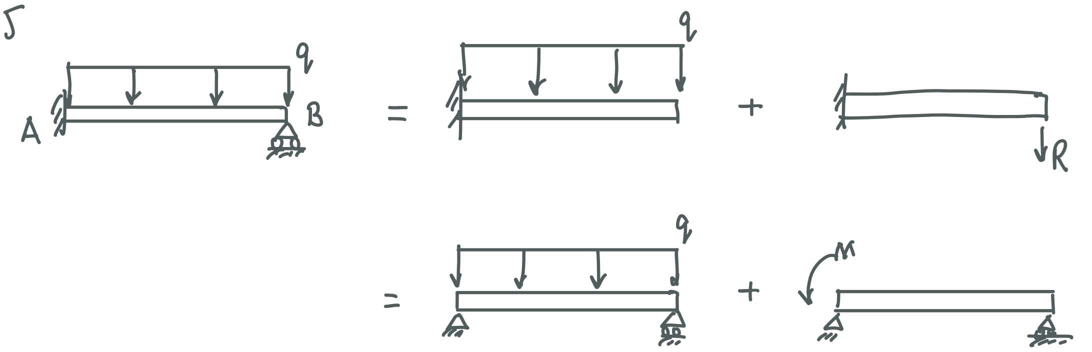
$$R_A = R_B = \frac{3}{16}ql$$

例4 叠加法




在C处, $v_q = \dfrac{ql^4}{384EI_z}$, $v_R = \dfrac{-Rl^3}{48EI_z}$, $v_q + v_R = 0 \rightarrow R = \dfrac{5}{8}ql.$

例5




(需使得 $v_B=0$)

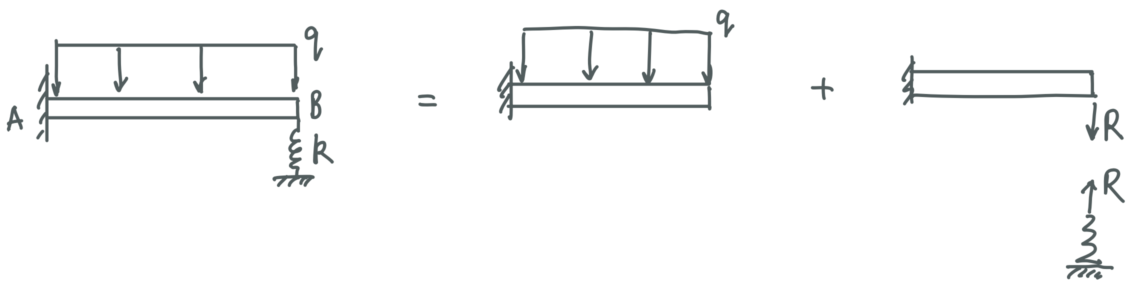
(需使得 $\theta_A=0$)

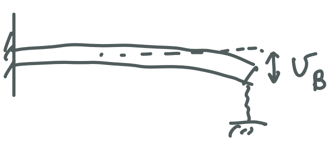
采用第一种叠加方法，在B处 $v_q=\frac{ql^4}{8EI_z}$，$v_R=\frac{Rl^3}{3EI_z}$

$$v_B=v_q+v_R=0 \to R=-\frac{3}{8}ql$$

负号说明真实方向与假设方向相反.

例6




$$v_B=v_q+v_R=\frac{ql^3}{8EI_z}+\frac{Rl^2}{3EI_z}$$

$$v_B=-\frac{R}{k}$$

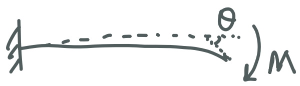
$$\longrightarrow R=\frac{-3ql^4}{8l^3+24EI_z/k}$$

Check: $k\to\infty$, → , $R\to-\frac{3}{8}ql$ ✓

$k\to\infty$，$R\to 0$

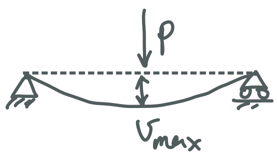
# § 5.4. 梁的刚度计算

## 刚度定义：

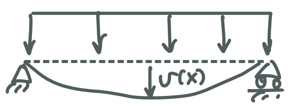



$$K = \frac{M}{\theta(l)} = \frac{EI_z}{l}$$

$EI_z\theta' = M \to EI_z\theta = Mx$




$$K = \frac{P}{v_{max}} = \frac{48EI_z}{l}$$




$$K = \frac{q}{\omega} = \frac{120EI_z}{l^5}$$

定义广义位移 $\omega = \int_0^l v\,dx = \frac{ql^5}{120EI_z}$ 为什么叫做广义位移?

## 刚度条件：

许用挠度 $\left[\frac{v}{l}\right] \sim \begin{cases} \frac{1}{250} - \frac{1}{1000} & \text{土建} \\ \frac{1}{5000} - \frac{1}{10000} & \text{机械制造} \end{cases}$

许用转角 $[\theta] \sim 0.005 - 0.001$ rad 传动轴

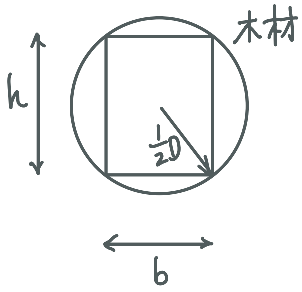
条件: $\frac{v_{max}}{l} \leq \left[\frac{v}{l}\right]$ , $\theta_{max} \leq [\theta]$

## 合理设计：

$$EIv'' = M \to v(x) = \left[v(0) + v'(0)x + \int_0^x\left(\int_0^x M\,dx\right)dx\right] / EI_z$$

显然，可以通过增加 $E$，$I_z$，减小 $l$，优化 $M(x)$ 来减小 $v_{max}$！

• 优化 $I_z$




李诫 (1100)：$h/b=\frac{3}{2}$

Young (1807)：$\frac{h}{b}=\sqrt{2}$ 时强度最大，$\frac{h}{b}=\sqrt{3}$ 时刚度最大。

$h^2+b^2=a^2$ （约束）

强度：$W=\frac{I_z}{y_{max}}=\frac{h^2 b}{6}$

$=\frac{1}{6}(D^2-b^2)b$

$\frac{\partial W}{\partial b}\Big|_R=\frac{1}{6}(D^2-b^2)-\frac{1}{3}b^2=0$

$\rightarrow b=\sqrt{\frac{1}{3}}D$，$h=\sqrt{\frac{2}{3}}D$

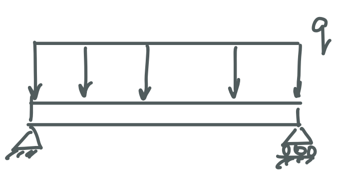
刚度：$I_z=\frac{1}{12}h^3 b$

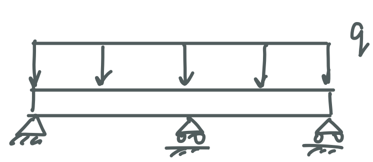
$=\frac{1}{12}(D^2-b^2)^{3/2}b$

$\frac{\partial I_z}{\partial b}=\frac{1}{12}(D^2-b^2)^{3/2}-\frac{1}{4}(D^2-b^2)^{1/2}\cdot b^2=0$

$\rightarrow b=\frac{1}{2}D$，$h=\frac{\sqrt{3}}{2}D$

• 减小跨长 $l$



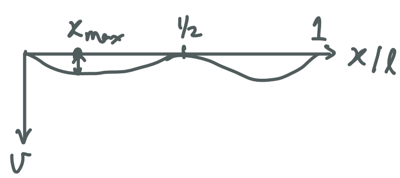




$\omega_2=\frac{ql^5}{5120EI_z}$

$v_{max}=\frac{5}{385}\frac{ql^4}{EI_z}$

$\omega_1=\frac{ql^5}{120EI_z}$

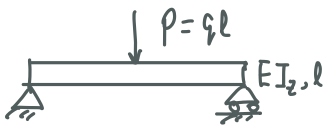
$$\boxed{\frac{K_2}{K_1}=\frac{\omega_2}{\omega_1}=42.7}$$

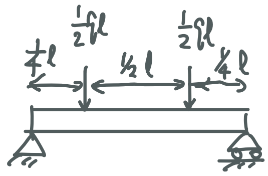


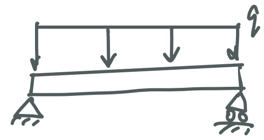

$x_{max}\simeq 0.2l$ or $0.8l$

$v_{2max}\simeq\frac{0.130}{385}\frac{ql^4}{EI_z}=\frac{1}{38.5}v_{1max}$

## · 优化载荷

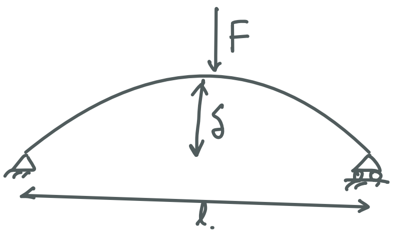



$$v_{max} = \frac{Pl^3}{48EI_z} = \frac{8ql^4}{384EI_z} \qquad v_{max} = \frac{5.5ql^4}{384EI_z} \qquad v_{max} = \frac{5ql^4}{384EI_z}$$

## · 预加反弯度



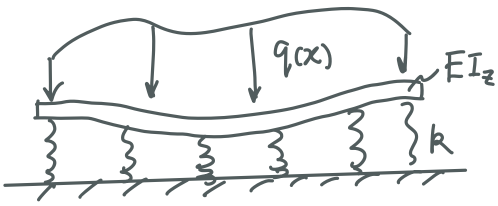

$$\frac{\delta}{l} \sim \frac{1}{700} - \frac{1}{500}$$

起拱（提前量，GB: $l > 4m$，必须设计起拱）

$$EI_z(v - v_0)'' = M$$

## · 增加等效 $I_z$

## §5.5. 弹性基础上梁的弯曲




Winkler (1867) 枕木在外力作用下的变形

将地基假设为一系列互不联系的弹簧

- Hertz 漂浮在水面上的冰层 $k=\rho g$
- Stotheim & Mahadevan (2004) 薄弹性层 $k=\frac{2(1-\nu)}{(1-2\nu)}\frac{G}{d}$ $d\sim$ 厚度

$$EI_z v'''' = q(x) + q_s(x)$$

$$q_s(x) = -kv$$

反力与挠度方向相反

$$\longrightarrow \boxed{EI_z v'''' + kv = q(x)}$$

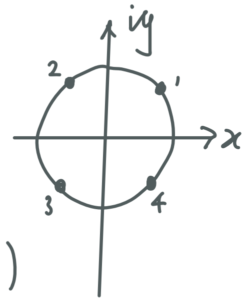
这是一个4阶，常系数线性微分方程

首先考查 $q(x)=0$ 时齐次方程的4个通解，其形式为 $v = Ae^{px}$

$$EI_z v'''' + kv = Ae^{px}(EI_z p^4 + k) = 0$$

$$\rightarrow p^4 = -k/EI_z = \frac{k}{EI_z} e^{i(2n+1)\pi}, \quad n = 1, 2, \cdots$$

$$\rightarrow p_1 = \left(\frac{k}{EI_z}\right)^{1/4} e^{i\frac{1}{4}\pi} = \underbrace{\left(\frac{k}{EI_z}\right)^{1/4}\frac{\sqrt{2}}{2}}_{\mu=\left(\frac{k}{4EI_z}\right)^{1/4}}(1+i)$$




$$p_2 = \mu(-1+i), \quad p_3 = \mu(-1-i), \quad p_4 = \mu(1-i)$$

因此，通解为 $v_H(x) = Ae^{\mu(1+i)x} + Be^{\mu(-1+i)x} + Ce^{-\mu(1+i)x} + De^{-\mu(-1+i)x}$

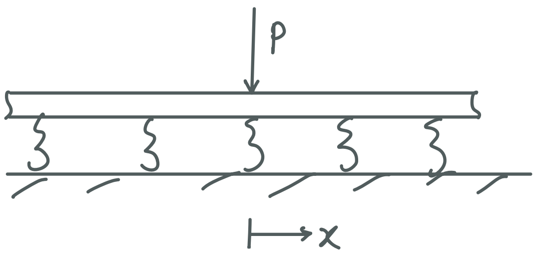
A, B, C, D 为复数使得 $v_H$ 为实数。亦可将通解写为

$$v_H(x) = e^{\mu x}(C_1 \sin\mu x + C_2 \cos\mu x) + e^{-\mu x}(C_3 \sin\mu x + C_4 \cos\mu x)$$

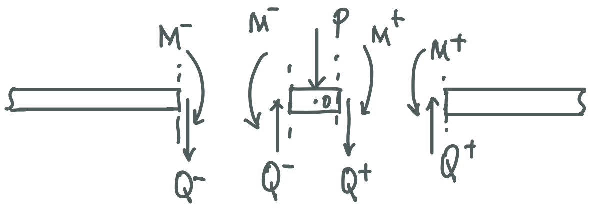
其中 $C_1, C_2, C_3, C_4$ 为实数。或利用双曲函数 $\cosh\mu x = \frac{e^{-\mu x}+e^{\mu x}}{2}$，$\sinh = \frac{e^{\mu x}-e^{-\mu x}}{2}$ 可写

$$v_H(x) = \cosh\mu x\,(D_1 \sin\mu x + D_2 \cos\mu x) + \sinh\mu x\,(D_3 \sin\mu x + D_4 \cos\mu x)$$

## 集中力解答




FBD




$\Sigma M_0 = 0 \rightarrow M^- = M^+$

$\Sigma F_y = 0 \rightarrow Q^+ + P = Q^-$

对称性（反转纸面）$\rightarrow Q^+ = -Q^-$

$\rightarrow -Q^+ = +Q^- = \frac{P}{2}$

考虑 $0<x<\infty$ 部分（另一部分对称）：$EI_z v'''' + kv \equiv 0$

$$\rightarrow v = e^{\mu x}(C_1 \sin\mu x + C_2 \cos\mu x) + e^{-\mu x}(C_3 \sin\mu x + C_4 \cos\mu x)$$

边界条件：

在 $x\rightarrow 0$ 处，$v'(0^+) = 0$，$EI_z v''' = -Q_y(0^+) = -Q^+ = \frac{P}{2}$

在 $x \rightarrow \infty$ 处，$v, v', v'', v''' \rightarrow 0 \longrightarrow C_1 = C_2 = 0$

$$v' = -\mu e^{-\mu x}(C_3 \sin\mu x + C_4\cos\mu x) + \mu e^{-\mu x}(C_3\cos\mu x - C_4\cos\mu x)$$

$$= \mu e^{-\mu x}\Big[-\overbrace{(C_3+C_4)}^{D_1}\sin\mu x + \overbrace{(C_3-C_4)}^{D_2}\cos\mu x\Big]$$

$$v'' = -\mu^2 e^{-\mu x}(-D_1\sin\mu x + D_2\cos\mu x) + \mu^2 e^{-\mu x}(-D_1\cos\mu x - D_2\sin\mu x)$$

$$= \mu^2 e^{-\mu x}\Big[\overbrace{(D_1-D_2)}^{E_1}\sin\mu x - \overbrace{(D_1+D_2)}^{E_2}\cos\mu x\Big]$$

$$v''' = \mu^3 e^{-\mu x}\Big[(-E_1+E_2)\sin\mu x + (E_1+E_2)\cos\mu x\Big]$$

代入边界条件：

$$\mu^4 = \frac{k}{4EI_z}$$

$$\left.\begin{aligned} &v'(0^+) = \mu(C_3-C_4) = 0 \to C_3 = C_4 \\ &v'''(0^+) = \mu^3(E_1+E_2) = \mu^3(2C_3+2C_4) = \frac{P}{2EI_z} \end{aligned}\right\} \to C_3 = C_4 = \frac{P}{8EI_z\mu^3} = \frac{\mu P}{2k}$$

$$\to v(x) = \begin{cases} \dfrac{\mu P}{2k} e^{-\mu x}(\sin\mu x + \cos\mu x), & 0 < x < \infty \\ \dfrac{\mu P}{2k} e^{+\mu x}(-\sin\mu x + \cos\mu x), & -\infty < x < 0 \end{cases}$$

注意：边界条件 $\frac{P}{2} = EI_z v'''(0^+) = EI_z v'''(0^+) - EI_z \overset{=0}{v'''(\infty)}$

$$= \int_\infty^0 EI_z v'''' \, dx$$

$$= \int_0^\infty k v \, dx$$ （物理含义：一半弹簧的合力＝一半的P）

因此，也可利用这一关系以及 $v'(0)=0$ 对应的 $C_3 = C_4$ 来求解，即

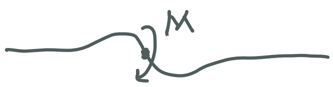
$$\frac{1}{2}P = \int_0^\infty k e^{-\mu x}(C_3\sin\mu x + C_4\cos\mu x)\,dx = -kC_3\frac{e^{-\mu x}}{\mu}\cos\mu x\Big|_0^\infty = \frac{kC_3}{\mu} \to C_3 = C_4 = \frac{\mu P}{2k}$$

# 集中力偶矩解答

平衡方程和解答思路相同，但采用不同的边界条件：

在 $x\to\infty$ 处， $v,\ v',\ v'',\ v''' \to 0 \quad \to C_1 = C_2 = 0$

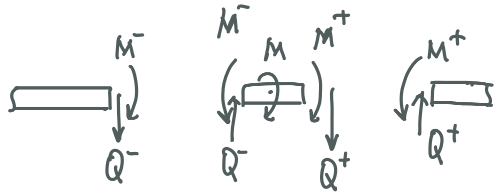
在 $x\to 0^+$ 处， $v\to 0$ ， $EI_z v'' \to ?\ \left(-\frac{1}{2}M\right)$




假设顺时针 M，导致 $v>0$

从纸的反面看，M为逆时针，$v>0$

$\to v(0)=0$




$Q^- = Q^+$

$$\left.\begin{aligned} M^- &= M + M^+ \\ M^- &= -M^+ \ (\text{对称}) \end{aligned}\right\} \to M^- = -M^+ = \frac{1}{2}M$$

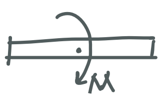
$v(0^+) = C_4 = 0$

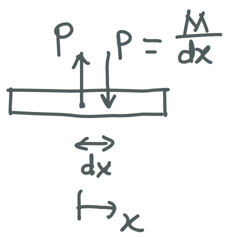
$$\mu^4 = \frac{k}{4EI_z}$$

$$v''(0^+) = \mu^2(-2C_3) = -\frac{1}{2}\frac{M}{EI_z} \to C_3 = \frac{M}{4\mu^2 EI_z} = \frac{\mu^2 M}{k}$$

$$\to v(x) = \begin{cases} \dfrac{\mu^2 M}{k} e^{-\mu x}\sin\mu x, & 0<x<\infty \\ \dfrac{\mu^2 M}{k} e^{\mu x}\sin\mu x, & -\infty<x<0 \end{cases}$$

反对称

另一个方法为 

= 


$$v_p = \frac{\mu P}{2k}\underbrace{e^{-\mu x}(\sin\mu x + \cos\mu x)}_{A(x)}$$

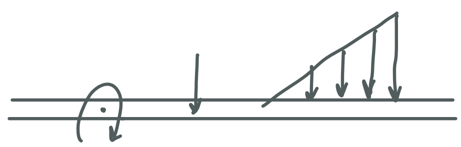
$$\sigma_M(x) = \lim_{dx\to 0}\left[-\frac{\mu P}{2k}A(x) + \frac{\mu P}{2k}\overbrace{A(x')}^{A(x-dx)}\right]$$

$$= \lim_{dx\to 0}\frac{\mu M}{2k}\,\frac{A(x-dx)-A(x)}{dx}$$

$$= -\frac{\mu M}{2k}\underbrace{\frac{dA(x)}{dx}}_{= -\mu e^{-\mu x}(\sin\mu x + \cancel{\cos\mu x}) + \mu e^{-\mu x}(\cancel{\cos\mu x} - \sin\mu x)}$$

$$= \frac{\mu^2 M}{k}e^{-\mu x}\sin\mu x$$

思考：



## §5.5. 常系数线性微分方程的初参数解法

$$\frac{d^n u}{dx^n} + a_1 \frac{d^{n-1} u}{dx^{n-1}} + \cdots + a_n u = f(x)$$

$$L u = f, \quad L = \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_n$$

$$L u_1 = f_1, \quad L u_2 = f_2 \rightarrow L\underbrace{(u_1 + u_2)}_{u} = L u_1 + L u_2 = \underbrace{f_1 + f_2}_{f}$$

方程通解 $u$ 为齐次方程 $Lu=0$ 的通解与特解的和.

- 先考虑 $Lu=0$ 的特解.

$n$阶 → 存在一组线性无关特解 $u_1, u_2, \cdots, u_n$

- 构造一组新的特解 $U_1, U_2, \cdots, U_n$

$$\begin{cases} U_k(0) = U_k'(0) = \cdots = U_k^{(n-1)}(0) = 0 \\ U_k^{(k-1)}(0) = 1 \end{cases}$$

即 $U_1(0)=1$, $U_2'(0)=1$, $U_n^{n-1}(0)=1$ ……

- 为实现上述目的,需

$$U_k = \sum_{i=1}^{n} C_{ki} u_i, \quad k = 1, 2, \cdots, n$$

$$U_k^{(s)}(0) = \sum_{i=1}^{n} C_{ki} u_i^{(s)}(0) = \delta_{s+1,k}, \quad s = 0, 1, \cdots, n-1$$

上式可具体写为:

$$\begin{bmatrix} u_1(0) & u_2(0) & \cdots & u_n(0) \\ u_1'(0) & u_2'(0) & \cdots & u_n'(0) \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(k-1)}(0) & u_2^{(k-1)}(0) & \cdots & u_n^{(k-1)}(0) \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-1)}(0) & u_2^{(n-1)}(0) & \cdots & u_n^{(n-1)}(0) \end{bmatrix} \begin{bmatrix} C_{k1} \\ C_{k2} \\ \vdots \\ C_{kk} \\ \vdots \\ C_{kn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$\begin{vmatrix} u_1(0) & & \\ & \ddots & \\ & & u_n^{(n-1)}(0) \end{vmatrix}$ 为函数系 $u_1, \cdots, u_n$ 在 $x=0$ 时的朗斯基行列式
Wronsky

$u_1, u_2, \cdots, u_n$ 线性无关 $\rightarrow$ 行列式 $\neq 0$ $\rightarrow$ 可解 $C_{ki}$

- 构造特解 $\underbrace{U_1(x), U_2(x), \cdots, U_n(x)}_{x=0,\text{朗斯基行列式}=1 \rightarrow \text{线性无关}}$，构成通解

$$u(x) = \sum_{i=1}^{n} C_i U_i(x)$$

- 研究非齐次方程 $Lu = f(x)$，可证明其特解为

$$\bar{u}(x) = \int_0^x U_n(x-\xi) f(\xi) d\xi$$

非齐次方程的通解为

$$\boxed{u(x) = \sum_{i=1}^{n} C_i U_i(x) + \int_0^x U_n(x-\xi) f(\xi) d\xi}$$

- 现确定 $C_i$：

① 令 $x=0$，$u(0) = C_1$

② 求一次导数 $u'(x)=\sum_{i=1}^{n}C_i U_i^{(1)}(x)+\cancel{U_n(0)}^{0}f(x)+\int_0^x U_n'(x-\xi)f(\xi)d\xi$

Leibniz $U_n(x-x)f(x)\frac{dx}{dx}$

参数量积分和微商定理

$u'(0)=C_2$

③ 求 k-1 次导数,得

$u^{(k-1)}(0)=C_k$

$$\Rightarrow \boxed{u(x)=\sum_{k=1}^{n}\underbrace{u^{(k-1)}(0)}_{\text{初值}}U_k(x)+\int_0^x U_n(x-\xi)f(\xi)d\xi}$$

初参数法

<u>挠曲方程:</u> $v''(x)=\frac{M_z}{EI_z}$

• 齐次方程 $v''=0$,通解:

$U_1(x)=1$, $U_2(x)=x$

• 非齐次方程

$v(x)=v(0)\cdot 1+v'(0)\cdot x+\int_0^x(x-\xi)\frac{M_z(\xi)}{EI_z}d\xi$

• 记 $\frac{M_z(\xi)}{EI_z}=\frac{d}{d\xi}\int_0^\xi \frac{M_z(\eta)}{EI_z}d\eta$

$v(x)=v(0)+v'(0)x+\int_0^x(x-\xi)\,d\int_0^\xi\frac{M_z(\eta)}{EI_z}d\eta\,d\xi$

$$\boxed{v(x)=v(0)+v'(0)x+\int_0^x\int_0^\xi\frac{M_z(\eta)}{EI_z}d\eta\,d\xi}$$

挠曲方程: $v''''(x) = \frac{q(x)}{EI_z}$

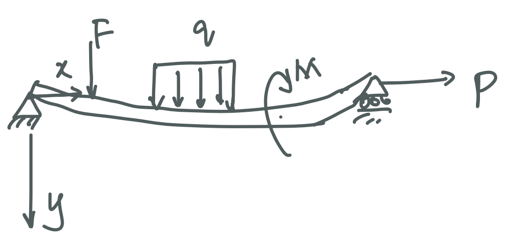
· 齐次方程 $v''''(x)=0$ 通解

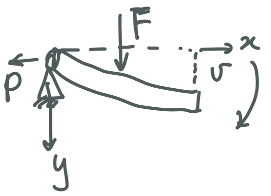
$$v_1=1,\quad v_2=x,\quad v_3=\frac{x^2}{2},\quad v_4=\frac{x^3}{6}$$

· 非齐次方程通解为

$$v(x) = v(0)+v'(0)x+\frac{1}{2}v''(0)x^2+\frac{1}{6}v'''(0)x^3+\int_0^x \frac{1}{6}(x-\xi)^3\frac{q(\xi)}{EI_z}d\xi$$

## § 5.6. 纵-横弯曲

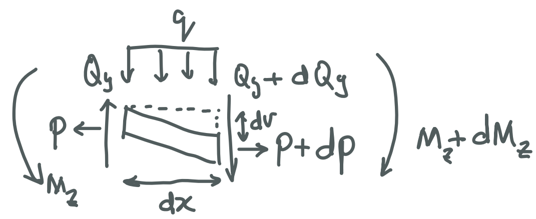



横向载荷

$$M_z = M_z^*(x) + Pv = EI_z v'' \quad (几何与本构)$$

或




$\Sigma F_x=0 \to \frac{dP}{dx}=0 \to P=$ 常数

$\Sigma F_y=0 \to \frac{dQ_y}{dx}=-q$

$\Sigma M=0 \to \frac{dM_z}{dx}=-Q_y+P\frac{dv}{dx}$

⇓

$$v''-\frac{P}{EI_z}v=\frac{M_z^*}{EI_z} \qquad 或 \qquad \frac{d^2M_z}{dx^2}=q+Pv'' \Rightarrow v''''-\frac{P}{EI_z}v''=\frac{q}{EI_z}$$

(1) $P>0$, 令

$$\frac{P}{EI_z}=k^2 \to v''-k^2v=0 \quad (齐次方程)$$

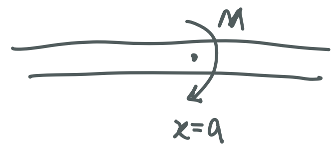
通解为 $\{e^{+kx}, e^{-kx}\}$ 或 $\left\{\cosh kx = \frac{e^{kx}+e^{-kx}}{2},\ \sinh kx = \frac{e^{kx}-e^{-kx}}{2}\right\}$

注意: $\cosh 0 = 1$, $\sinh 0 = 0$, $\cosh' kx = k \sinh kx$, $\sinh' kx = k\cosh kx$

$$U_1 = \cosh kx, \quad U_2 = \frac{1}{k}\sinh kx$$

$$\Rightarrow v(x) = v(0)\cosh kx + v'(0)\frac{1}{k}\sinh kx + \int_0^x \underbrace{\frac{1}{k}\sinh k(x-\xi)\frac{M_z^*(\xi)}{EI_z}}_{\text{考查}} d\xi$$

① 在 $x=a$ 处作用力偶矩 $M$，弯矩

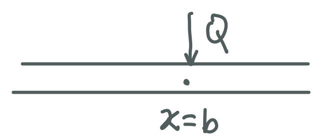



$$M_z^*(x) = -M\varphi_0(x-a) = \begin{cases} 0, & x<a \\ -M, & x>a \end{cases}$$

$$\int_0^x \sinh k(x-\xi)\varphi_0(\xi-a)d\xi = \int_a^x \sinh k(x-\xi)\,d\xi = \frac{-1}{k}\cosh k(x-\xi)\Big|_a^x$$

$$\boxed{= \frac{1}{k}\left[\cosh k(x-a) - 1\right]}$$

② 在 $x=b$ 处作用集中力 $Q$，




$$M_z^*(x) = Q\varphi_1(x-b) = \begin{cases} 0, & x<b \\ Q(x-b), & x>b \end{cases}$$

$$\int_0^x \sinh k(x-\xi)\varphi_1(\xi-b)d\xi = \int_b^x \sinh k(x-\xi)(\xi-b)\,d\xi$$

$$= -\left[\frac{1}{k^2}\sinh k(x-\xi) + \frac{1}{k}(\xi-b)\cosh k(x-\xi)\right]_b^x$$

$$\boxed{= \frac{1}{k^2}\sinh k(x-b) - \frac{x-b}{k}}$$

$$\Rightarrow\ v(x) = v(0)\cosh kx + v'(0)\frac{1}{k}\sinh kx$$

$$\underbrace{+\frac{1}{EI_z}\sum\left\{-\frac{M}{k^2}\left[\cosh k(x-a) - 1\right] + Q\left[\frac{1}{k^3}\sinh k(x-b) - \frac{x-b}{k^2}\right]\right\}}_{\parallel}$$

$$\frac{1}{EI_z} = \frac{k^2}{P}$$

$$\frac{1}{P}\sum\left\{-M\left[\cosh k(x-a) - 1\right] + Q\left[\frac{1}{k}\sinh k(x-b) - (x-b)\right]\right\}$$

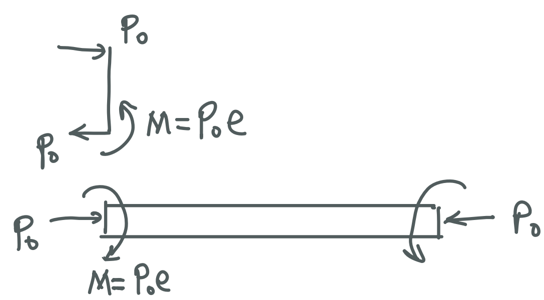
(2) $P<0$ , $\hat{2}$

$$k^2 = -\frac{P}{EI_z}\ ,\quad v'' + k^2 v = \frac{M_z^*(x)}{EI_z}$$

$$\Rightarrow\ U_1(x) = \cos kx\ ,\quad U_2(x) = \frac{1}{k}\sin kx$$

$$\Rightarrow\ \boxed{\begin{aligned} v(x) &= v(0)\cos kx + v'(0)\frac{1}{k}\sin kx \\ &-\frac{1}{P}\sum\left\{-M\left[1-\cos k(x-a)\right] + Q\left[(x-b) - \frac{1}{k}\sin k(x-b)\right]\right\}\end{aligned}}$$

例7


$$P = -P_0 < 0\ ,\quad M = P_0 e\ ,\quad \frac{M}{P} = -e$$

$$v(x) = \frac{v'(0)}{k}\sin kx - e(1-\cos kx) \qquad \left[\text{here } v(0)=0\ ,\ \varphi_0(x-a) \text{ 中取 } a=0\right]$$

$$v(l) = \frac{v'(0)}{k}\sin kl - e(1-\cos kl) = 0 \ \rightarrow\ v'(0) = \frac{ek(1-\cos kl)}{\sin kl}$$

$$\Rightarrow \boxed{v(x)=e\left[\frac{1-\cos kl}{\sin kl}\sin kx-(1-\cos kx)\right],\quad k=\sqrt{\frac{P_0}{EI_z}}}$$

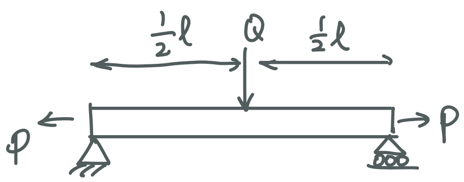
$kl=\sqrt{\dfrac{P_0 l^2}{EI_z}}=n\pi\ (n=1,2,\cdots)$ 时 $v\to+\infty$ 为什么?

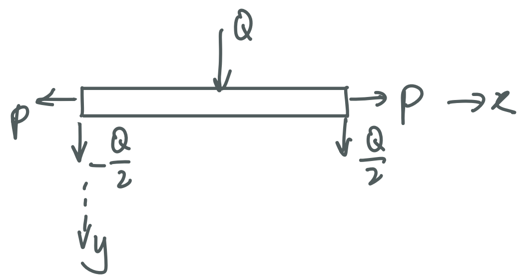
(b) 若 $P=P_0>0$ (拉力)

$$\Rightarrow \boxed{v(x)=-e\left[\frac{\cosh kl-1}{\sinh kl}\sinh kx-(\cosh kx-1)\right],\quad k=\sqrt{\frac{P_0}{EI_z}}}$$

$kl=\sqrt{\dfrac{P_0 l^2}{EI_z}}\to\infty$, $\cosh kl\to\sinh kl$, $v(\frac{l}{2})\to -e$

例8




对称性，只考虑 $0\leq x\leq \frac{1}{2}l$ 部分

$$\Rightarrow v(x)=v'(0)\frac{1}{k}\sinh kx+\frac{-Q}{2P}\left(\frac{1}{k}\sinh kx-x\right)$$

$$v'(\tfrac{1}{2}l)=v'(0)\cosh\tfrac{1}{2}kl-\frac{Q}{2P}\left(\cosh\tfrac{1}{2}kl-1\right)=0\leftarrow \text{对称性}$$

$$v'(0)=\frac{Q}{2P}\left(1-\frac{1}{\cosh(kl/2)}\right)$$

$$\Rightarrow v(x)=-\frac{Q}{2Pk}\frac{\sinh kx}{\cosh kl/2}+\frac{Q}{2P}x,\quad k=\left(\frac{P}{EI_z}\right)^{1/2}$$

可知 


$$M(x) = P v(x) - \frac{1}{2} Q x = -\frac{Q}{2k} \frac{\sinh kx}{\cosh kl/2}$$

对称性可知 $M_{max} = M(\frac{l}{2}) = -\frac{Q}{2k} \tanh kl/2$

最大正应力 $\sigma_{max} = \frac{|M_{max}|}{W} + \frac{P}{A}$

抗弯截面系数　　横截面面积

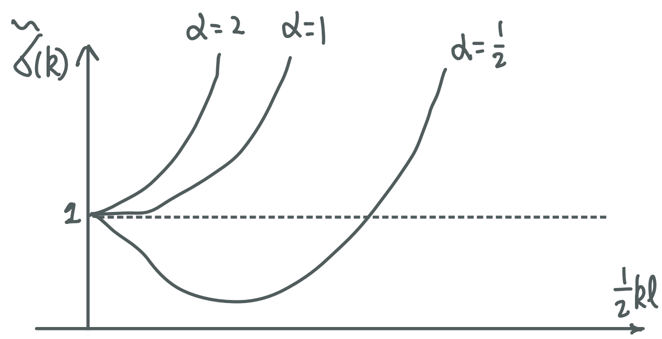
$$\sigma(k) = \underbrace{\frac{Ql}{4W} \frac{\tanh(kl/2)}{kl/2}}_{\sigma_1(k)} + \underbrace{\frac{EI_z}{A} k^2}_{\sigma_2(k)}$$

$k=0$, $\sigma(0) = \frac{Ql}{4W}$ （由横力 F 引起）

$$\tilde{\sigma}(k) = \frac{\sigma(k)}{\sigma(0)} = \underbrace{\frac{\tanh(kl/2)}{kl/2}}_{\tilde{\sigma}_1} + \underbrace{\frac{1}{3}\alpha(kl/2)^2}_{\tilde{\sigma}_2}, \quad \alpha = \frac{48WEI_z}{QAl^3}$$

- $\tilde{\sigma}_1 = \begin{cases} 1 - \frac{1}{3}(\frac{1}{2}kl)^2 + \frac{2}{15}(\frac{1}{2}kl)^4, & kl \to 0 \\ 0, & kl \to \infty \end{cases}$
- $\tilde{\sigma}_2 = \frac{1}{3}\alpha(\frac{1}{2}kl)^2$

P↑, k↑, 拉直, $\tilde{\sigma}$ 由 $\tilde{\sigma}_2$ 主导




① $\alpha \geq 1$, $\sigma(k) \geq 1$ $(\forall\ kl > 0)$

② $\alpha < 1$, $\sigma(k)$ 可小于 1

（一定的轴力可改善梁的安全程度）