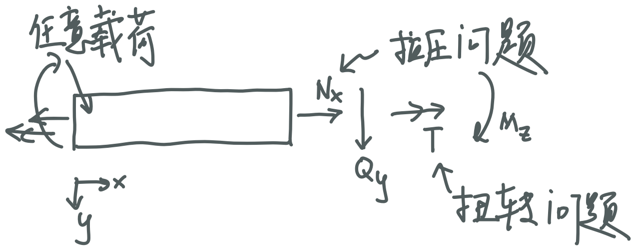
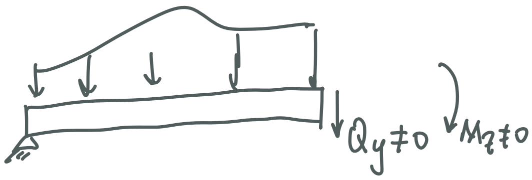
第四章 弯曲内力与弯曲应力



还有剪力Q，弯矩M_z尚未讨论.



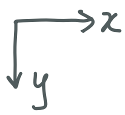
弯曲：外力垂直于轴线，轴线变形为曲线

梁：以弯曲变形为主的杆件.

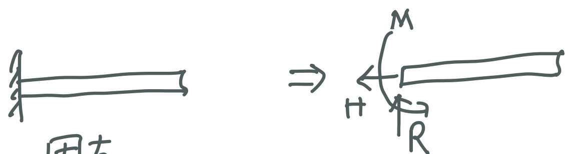
§4.1 剪力&弯矩

首先我们确定一下关于Q&M的符号约定.(方便讨论后续问题)

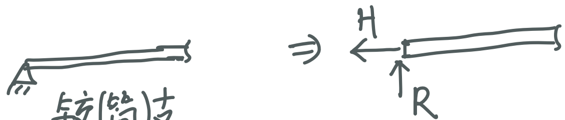
坐标系：



约束：



固支

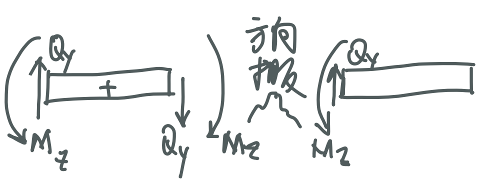


铰(简)支

可动铰支

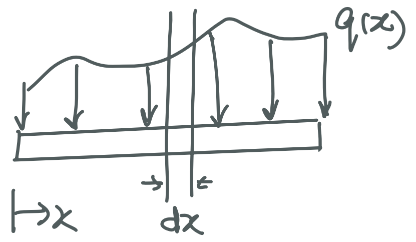
滑动支座

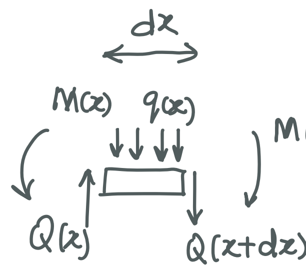
正方向：



确保同一截面，沿±x的内力符号统一.

外力，剪力，弯矩的微分关系（平衡方程）





$$M(x+dx) = M(x) + \frac{dM}{dx}dx + O(dx^2)$$

$$Q(x+dx) = Q(x) + \frac{dQ}{dx}dx + O(dx^2)$$

$$\Sigma F_y = -\cancel{Q(x)} + \left[\cancel{Q(x)} + \frac{dQ}{dx}dx + O(dx^2)\right] + q(x)\,dx + \underbrace{O(dx^2)}_{q\text{在}dx\text{内的变化}} = 0$$

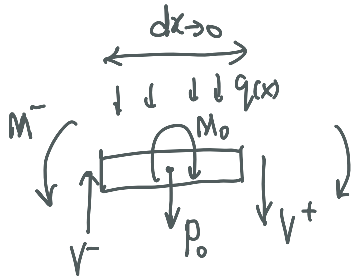
$$\lim_{dx\to 0} \to \boxed{\frac{dQ}{dx} = -q(x)}$$

$$\Sigma M_z^x = -\cancel{M(x)} + \cancel{M(x)} + \frac{dM}{dx}dx + O(dx^2) + \left[Q(x) + \frac{dQ}{dx}dx + O(dx^2)\right]\cdot dx$$

$$+\, q(x)\cdot dx \cdot \frac{1}{2}dx + O(dx^3) = 0$$

$$\lim_{dx\to 0} \to \boxed{\frac{dM}{dx} = -Q(x) \quad \text{或} \quad \frac{d^2M}{dx^2} = q(x)}$$

有时存在集中力和集中力矩，此时Q, M可能会发生"间断"或"跳跃"，例如：

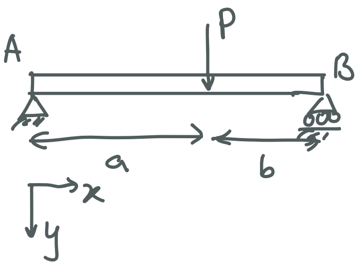


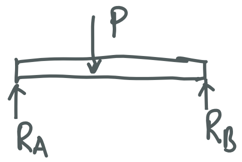
$$\Sigma F_y = 0 \to V^+ - V^- = -P_0 + (H.O.T.^{\sim q} \to 0)$$

$$\Sigma M_z = 0 \to M^+ - M^- = -M_0 + (H.O.T.^{\sim q, V} \to 0)$$

这些微分关系和跳跃关系足以确定梁中剪力和弯矩的分布情况！

例1 简支梁（一端固定铰支，另一端可动铰支）



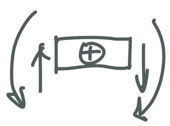


$\Sigma F_y = R_A + R_B - P = 0$

$\Sigma M_z^A = P \cdot a - (a+b) R_B = 0$

$\rightarrow$ $R_A = \frac{b}{l} P$

$R_B = \frac{a}{l} P.$

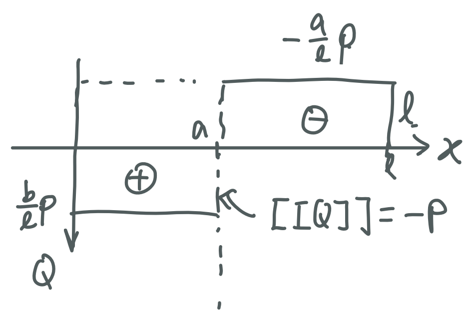


$Q(x=0) = \frac{b}{l} P$ ， $Q(x=l) = -\frac{a}{l} P$

$M(x=0) = 0$ ， $M(x=l) = 0$

在 $0 \le x < a$ 段， $\frac{dQ}{dx} = 0 \rightarrow Q = \frac{b}{l} P.$

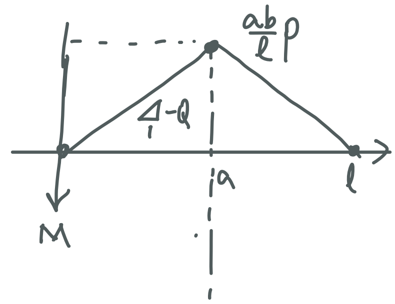
$\frac{dM}{dx} = -Q \rightarrow M = \frac{b}{l} P x$



在 $a < x \le l$ 段， $\frac{dQ}{dx} = 0 \rightarrow Q = -\frac{a}{l} P$

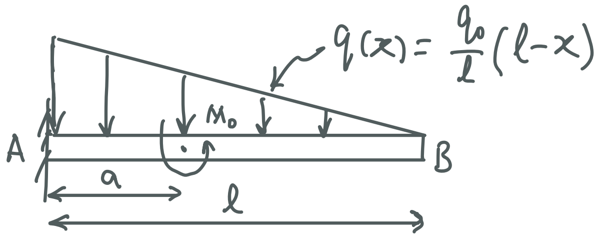
$\frac{dM}{dx} = Q \rightarrow M = \frac{a}{l} P (l - x)$

满足 $M(l) = 0$

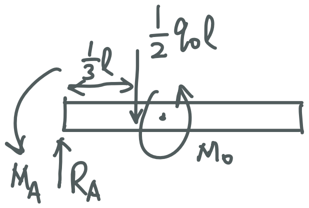


Check： $M(a^+) = \frac{a}{l} P (l - a) = \frac{ab}{l} P$ ✓.

例2 悬臂梁（一端固支，另一端自由）



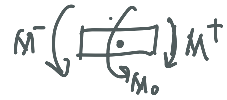
求约束反力可等价



$\Sigma F_y = -R_A + \frac{1}{2}q_0 l = 0 \rightarrow R_A = \frac{1}{2}q_0 l$

$\Sigma M_z^A = -M_A + \frac{1}{6}q_0 l^2 - M_0 = 0 \rightarrow M_A = -M_0 + \frac{1}{6}q_0 l^2$

$\rightarrow Q(0) = \frac{1}{2}q_0 l, \quad V(l) = 0$

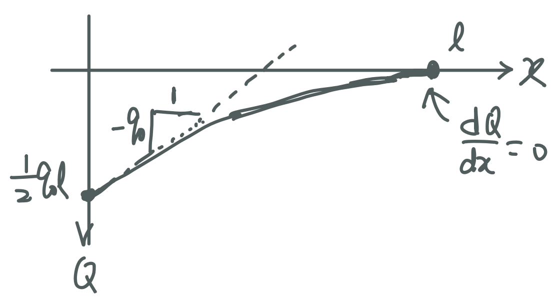


$M(0) = -M_0 + \frac{1}{6}q_0 l^2, \quad M(l) = 0$ （注意 $M(a^+) - M(a^-) = M_0$）

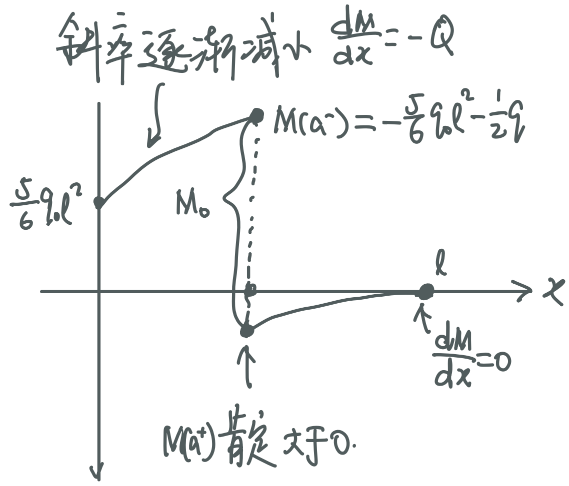
• $Q(x)$连续光滑

$$\frac{dQ}{dx} = -\frac{q_0}{l}(l-x), \quad 0 \le x \le l$$

$$\rightarrow Q(x) = \underbrace{\frac{1}{2}q_0 l}_{Q(0)} - q_0 x + \frac{q_0}{2l}x^2$$



• $M(x)$存在跳跃（让$M_0 = q_0 l^2$）



$$0 \le x < a \quad \frac{dM}{dx} = -\frac{1}{2}q_0 l + q_0 x - \frac{1}{2}q_0\frac{x^2}{l}$$

$$M(x) = \underbrace{-M_0 + \frac{1}{6}q_0 l^2}_{-\frac{5}{6}q_0 l^2} - \frac{1}{2}q_0 l x + \frac{1}{2}q_0 x^2 - \frac{1}{6}q_0\frac{x^3}{l}$$

$$a < x \le l \quad \frac{dM}{dx} = -\frac{1}{2}q_0 l + q_0 x - \frac{1}{2}q_0\frac{x^2}{l}$$

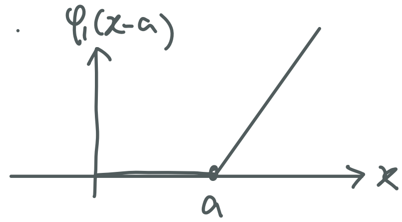
$$M(x) = -\frac{1}{2}q_0 l x + \frac{1}{2}q_0 x^2 - \frac{1}{6}q_0\frac{x^3}{l} + \underbrace{\frac{1}{6}q_0 l^2}_{\text{使得} M(l)=0}$$

Check: $M(a^+) - M(a^-) = q_0 l^2$ ✓

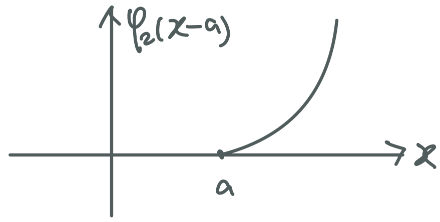
如何统一的解答 $\frac{d^2M}{dx^2} = q(x)$? — 特殊函数.

特殊函数 (Half-range / discontinuity functions)

定义 $\varphi_1(x-a)=\begin{cases}0, & x<a\\ x-a, & x>a\end{cases}$



$\varphi_2(x-a)=\begin{cases}0, & x<a\\ \frac{1}{2}(x-a)^2, & x>a\end{cases}$



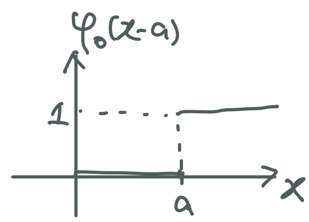
显然：$\int_0^x \varphi_1\,dx=\varphi_2$ 或 $\frac{d\varphi_2}{dx}=\varphi_1$

同样，我们可以定义 $\varphi_n(x-a)=\begin{cases}0, & x<a\\ \frac{1}{n!}(x-a)^n, & x>a\end{cases}$, $n=1,2,3,\ldots\ldots$

其具有性质：$\int_0^x \varphi_{n-1}\,dx=\varphi_n$ 或 $\frac{d\varphi_n}{dx}=\varphi_{n-1}$

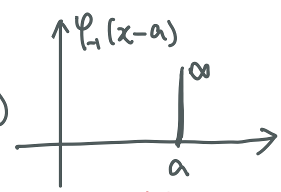
如何定义 $\varphi_0(x-a)$？ $\varphi_0(x-a)=\frac{d\varphi_1(x-a)}{dx}=\begin{cases}0, & x<a\\ 1, & x>a\end{cases}$

- $\int_0^x \varphi_0(x-a)\,dx=\begin{cases}0, & x<a\\ x-a, & x>a\end{cases}=\varphi_1(x-a)$



单位跃阶函数

同样地，我们定义 $\varphi_{-1}(x-a)=\frac{d\varphi_0(x-a)}{dx}=\begin{cases}0, & x\neq a\\ \infty, & x=a\end{cases}=\delta(x-a)$

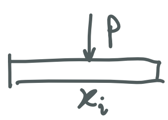


脉冲

- $\int_{a-\epsilon}^{a+\epsilon}\varphi_{-1}(x-a)\,dx=\int_{a-\epsilon}^{a+\epsilon}\frac{d\varphi_0(x-a)}{dx}\cdot dx=\varphi_0(x-a)\Big|_{a-\epsilon}^{a+\epsilon}=1$ for $\forall\epsilon>0$

- $\int_0^x \varphi_{-1}(x-a)\,dx=\begin{cases}0 & x<a\\ 1 & x>a\end{cases}=\varphi_0(x-a)$

- φ_{-1} 可用于表示集中力



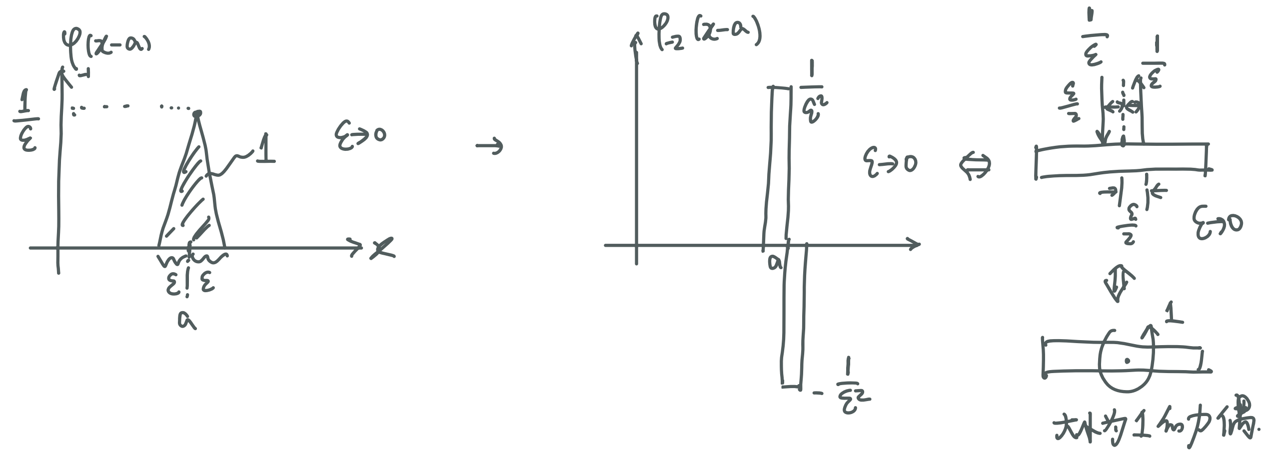
$q(x)=P\varphi_{-1}(x-x_i)$

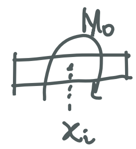
Check: ① $\int_{x_i-\varepsilon}^{x_i-\varepsilon} q(x)\,dx = P$ ✓ 分布力在无限小区域积分恒为 P

② $\frac{dQ}{dx} = -q = -P\varphi_{-1}(x-x_i) \rightarrow Q = -P\varphi_0(x-x_i) + C_1 \rightarrow Q(x_i^+) - Q(x_i^-) = -P$ ✓

最后一个函数定义 $\varphi_{-2}(x-a) = \frac{d\varphi_{-1}(x-a)}{dx}$ (也就是 $\delta'(x-a)$)，使得 $\int_0^x \varphi_{-2}(x-a)\,dx = \varphi_{-1}(x-a)$

首先可以将 $\varphi_{-1}(x-a)$ 想象为



Check: 

$q = -M_0\varphi_{-2}(x-x_i)$

$Q = \int_0^x -q\,dx = +M_0\varphi_{-1}(x-x_i) = \begin{cases} 0 & x \neq x_i \ ✓ \\ \boxed{\infty \quad x = x_i} \end{cases}$ 并无实际意义

$M = \int_0^x -Q\,dx = -M_0\varphi_0(x-x_i) = \begin{cases} 0 & x < x_i \\ -M_0 & x > x_i \end{cases}$

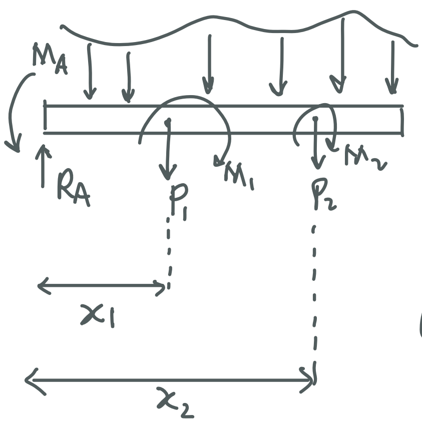
$M(x_i^+) - M(x_i^-) = -M_0$ ✓

总结：① 对所有 $\varphi_n(x-a)$，$\int_0^x \varphi_n(x-a)\,dx = \varphi_{n+1}$，$n = -2, -1, 0, 1, \cdots$

② 集中力 P↓ 的分布力形式为 $q(x) = P\varphi_{-1}(x-x_i) = P\delta(x-x_i)$

集中力偶 M_0(↻) 的分布力 $q(x) = -M_0\varphi_{-2}(x-x_i) = -M_0\delta'(x-x_i)$

③ $\varphi_n(x-a) = \begin{cases} 0, & x<a \\ \frac{1}{n!}(x-a)^n, & x>0 \end{cases}$，$n = 0, 1, 2, \cdots$

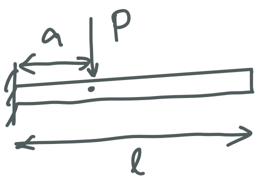


$\bar{q}(x)$ "常规"分布载荷

$$q(x)=\bar{q}(x)+\sum_i P_i\varphi_{-1}(x-x_i)-\sum_i M_i\varphi_{-2}(x-x_i)$$

$$Q(x)=\int_0^x -q(x)\,dx=-\int_0^x \bar{q}(x)\,dx-\sum_i P_i\varphi_0(x-x_i)+\sum_i M_i\varphi_{-1}(x-x_i)+\underbrace{Q(0)}_{R_A}$$

$$M(x)=\int_0^x -Q(x)\,dx=\int_0^x\left(\int_0^x \bar{q}(x)\,dx\right)dx+\sum_i P_i\varphi_1(x-x_i)-\sum_i M_i\varphi_0(x-x_i)-\underbrace{Q(0)}_{R_A}x+\underbrace{M(0)}_{M_A}$$



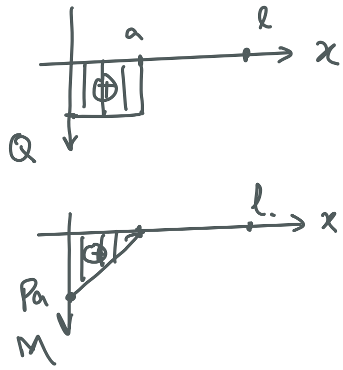
$\bar{q}(x)\equiv 0$，可立即得出

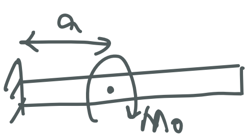
$$Q(x)=P-P\varphi_0(x-a)$$

$$M(x)=Pa+P\varphi_1(x-a)-Px$$

$R_A=P$

$M_A=Pa$



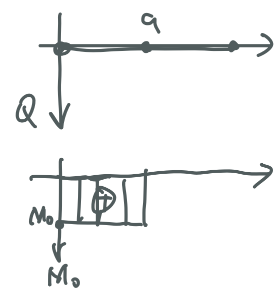


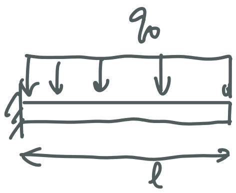
$$Q(x)=0$$

$$M(x)=M_0-M_0\varphi_0(x-a)$$

$R_A=0$

$M_A=M_0$

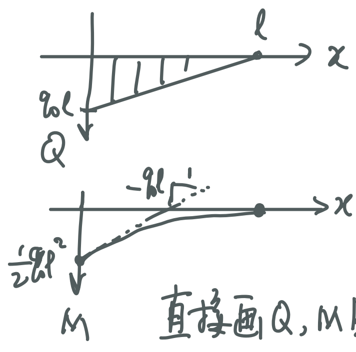




$$Q(x)=q_0l-q_0x$$

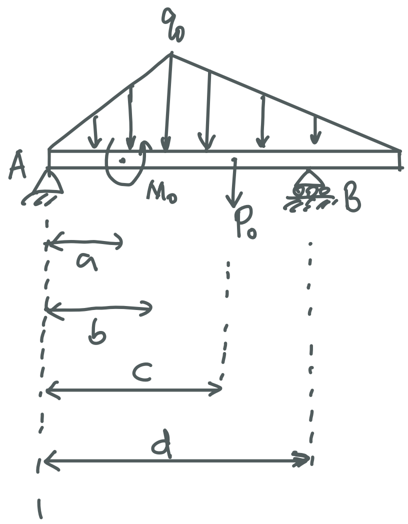
$$M(x)=\frac{1}{2}q_0l^2-q_0lx-\frac{1}{2}q_0x^2$$

$R_A=q_0l$, $M_A=\frac{1}{2}q_0l^2$

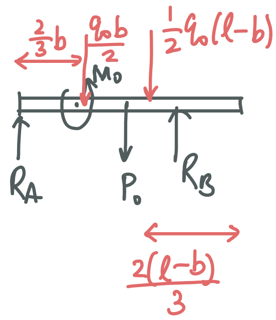


直接画Q, M图更为简单

例3 复杂载荷时，公式更为方便.



首先确定约束反力



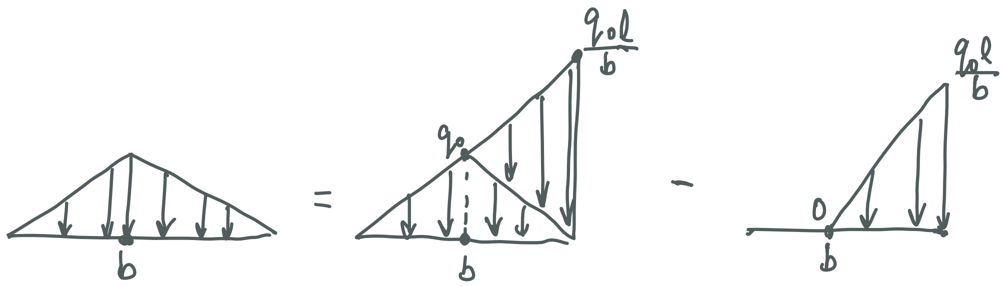
$$\sum F_y = -R_A - R_B + P_0 + \frac{q_0 b}{2} + \frac{q_0}{2}(l-b) = -R_A - R_B + P_0 + \frac{1}{2}q_0 l = 0$$

$$\sum M_z^A = M_0 - \frac{1}{3}q_0 b^2 - P_0 c - \frac{l+2b}{3}\cdot\frac{1}{2}q_0(l-b) + R_B \cdot d = 0$$

$$\rightarrow R_B = -\frac{M_0}{d} + \frac{P_0 c}{d} + \frac{1}{6}\frac{q_0 l^2}{d} + \frac{1}{6}\frac{q_0 l b}{d}$$

$$\rightarrow R_A = -R_B + P_0 + \frac{1}{2}q_0 l$$

然后确定 $q(x)$.



$$q_1(x) = \frac{q_0}{b}x \qquad q_2(x) = \frac{q_0 l}{b(l-b)}\varphi_1(x-b)$$

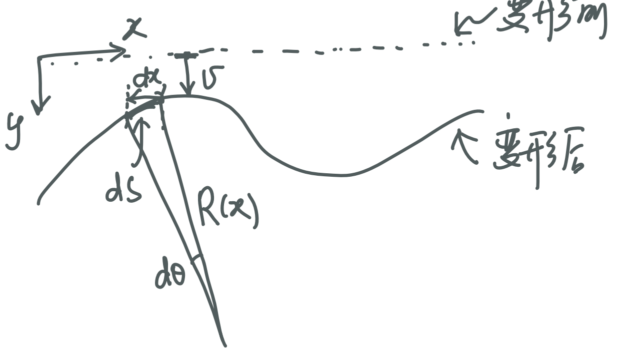
$$\Rightarrow q(x) = \frac{q_0}{b}x - \frac{q_0 l}{b(l-b)}\varphi_1(x-b) \underbrace{+ M_0}_{\text{逆时针}} \varphi_{-2}(x-a) + P_0\varphi_{-1}(x-c) - R_B\varphi_{-1}(x-d)$$

$$Q(x) = -\frac{q_0}{2b}x^2 + \frac{q_0 l}{b(l-b)}\varphi_2(x-b) - M_0\varphi_{-1}(x-a) - P_0\varphi_0(x-c) + R_B\varphi_0(x-d) + R_A$$

$$M(x) = \frac{q_0}{6b}x^3 - \frac{q_0 l}{b(l-b)}\varphi_3(x-b) + M_0\varphi_0(x-a) + P_0\varphi_1(x-c) - R_B\varphi_1(x-d) + R_A x + \cancel{M_A}^{=0}$$

§4.2. 弯曲正应力

现在，我们讨论弯曲问题的几何。——如何刻画弯曲变形？



v：向下（+y）的位移 / 挠度

R：变形的曲率半径（⊕ ⊖）

dx：微元长度（变形前）

ds：微元弧长（变形后）

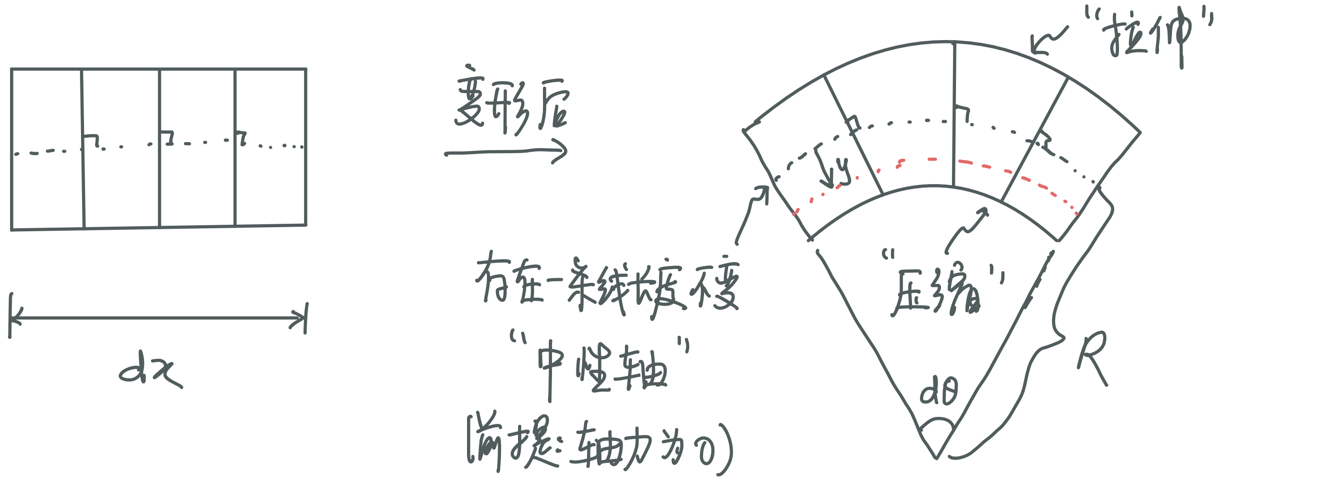
$d\theta$：由 ds 扫过的角度

$$ds = R\,d\theta \quad \text{或} \quad \underbrace{\frac{1}{R}}_{\text{曲率}} = \frac{d\theta}{ds}$$

下一章讨论 θ, v, R 之间的关系，这里先认为 v, θ 都很小，$ds \simeq dx$，也就是

$$\frac{1}{R} \simeq \frac{d\theta}{dx}$$

根据观察，采用平截面假设：垂直于轴的平截面在变形后仍为平面，且与轴垂直



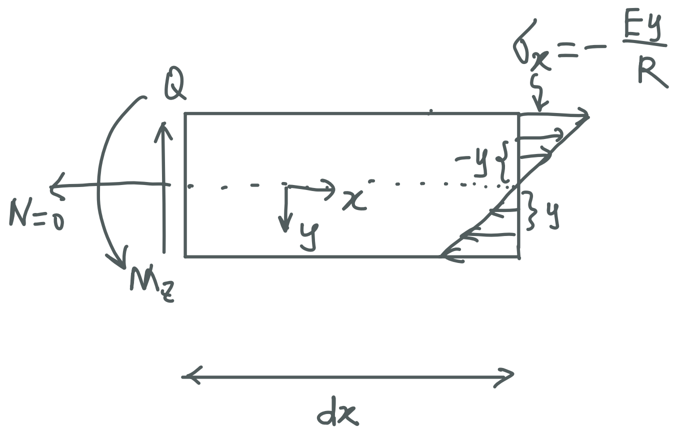
· 中性轴：$R \cdot d\theta = dx$ （长度不变）

· 距中性轴 y 的线应变：$\varepsilon_x(y) = \dfrac{(R-y)d\theta - dx}{dx} = -\dfrac{y\,d\theta}{dx} = -\dfrac{y}{R}$

· 胡克定理：$\sigma_x(y) = E\varepsilon_x = -\dfrac{Ey}{R}$

负号的根源在于正的弯矩/曲率方向约定，以及 $\underset{y}{\downarrow}\!\!\xrightarrow{x}$ 坐标

· 平衡关系：



$$\Sigma F_x = \int_A \sigma_x \, dA = \underbrace{\boxed{-\frac{E}{R}\int_A y\,dA = 0}}$$

∴ 中性轴（$y=0$）建立于形心上.

$$\Sigma F_y = \underset{\text{通常知}}{Q} + \int_A \underset{?\ \text{下一节再讲}}{\tau} \, dA = 0$$

$$\Sigma M_z = M_z - \underbrace{\int_{A_-} -\frac{Ey}{R}\cdot(-y)\,dA - \int_{A_+} \frac{Ey}{R}\cdot y\,dA}_{= \frac{E}{R}\int_A y^2 dA = \frac{E}{R} I_z} = 0 \rightarrow \boxed{M_z = EI_z \frac{1}{R(x)} = EI_z \kappa(x)}$$

I_z：截面对于 z 轴的惯性矩.

Recall:

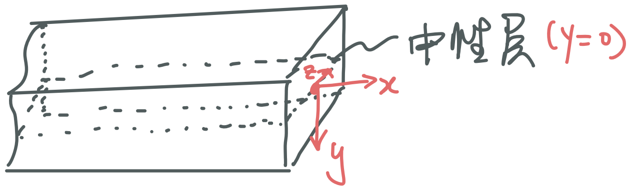
· $N = EA\,\varepsilon(x)$ 拉压

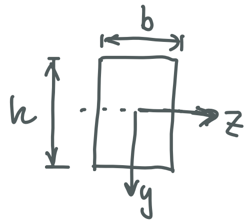
· $M_x = GI_p\,\theta(x)$ 扭转

$$\rightarrow \boxed{\begin{array}{l} \sigma_x = -\dfrac{M_z y}{I_z} \\ \text{或} \\ \sigma_x(x,y) = -\dfrac{M_z(x)\, y}{I_z(x)} \end{array}}$$

Recall: $\tau_\rho(x,\rho) = \dfrac{M_x \rho}{I_p}$ 扭转问题

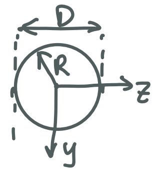
$\sigma_x(x) = \dfrac{N(x)}{A(x)}$ 拉压问题





矩形截面

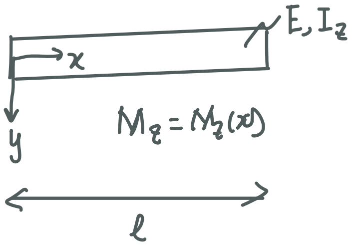
$$I_z=\int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 b\,dy = \frac{bh^3}{12}$$



圆形截面

$$I_z=\frac{\pi}{4}R^4=\frac{\pi}{64}D^4=\frac{1}{2}I_p$$

梁的弯曲变形能



$$U=\int_V \frac{1}{2}\sigma_x\varepsilon_x\,dV=\int_V\frac{1}{2}E\varepsilon_x^2\,dV=\int_V\frac{1}{2}\frac{\sigma_x^2}{E}\,dV \quad (\text{单位体积})$$

$$=\int_0^l\int_A\frac{1}{2}\frac{1}{E}\cdot\frac{M_z^2y^2}{I_z^2}\,dA\,dx$$

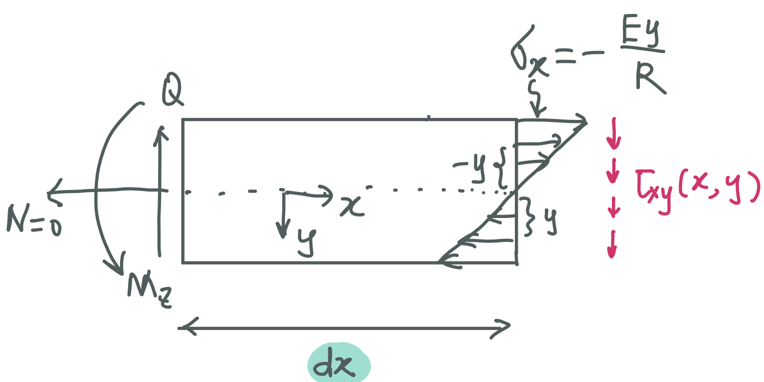
$$=\int_0^l\frac{M_z^2}{2EI_z}\underbrace{\int_A\frac{y^2}{I_z}\,dA}_{=1}\,dx$$

$$=\int_0^l\frac{M_z^2}{2EI_z}\,dx$$

Recall: $U=\int_V\frac{1}{2}\sigma_x\varepsilon_x\,dV=\int_0^l\frac{N^2}{2EA}dx$ 拉压问题

$U=\int_V\frac{1}{2}\tau\gamma\,dV=\int_0^l\frac{M_x^2}{2GI_p}dx$ 扭转问题

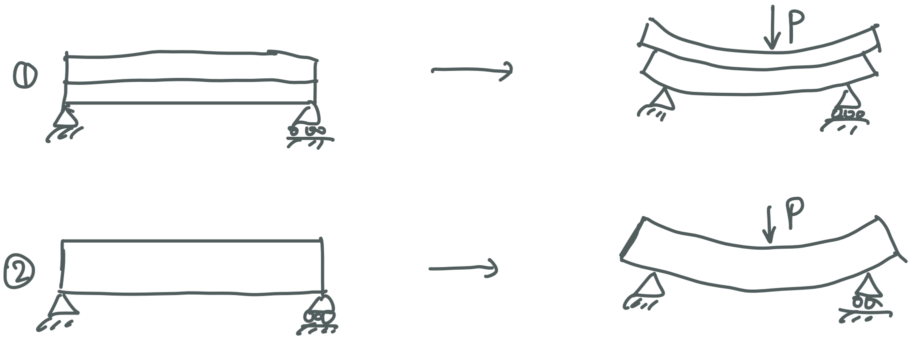
§4.3. 弯曲切应力



$\Sigma F_y=0 \rightarrow Q\neq0$ 时，存在切应力

$Q=0$ 时，$\tau_{xy}=0$ (将会证明) → 纯弯曲

生活经验:

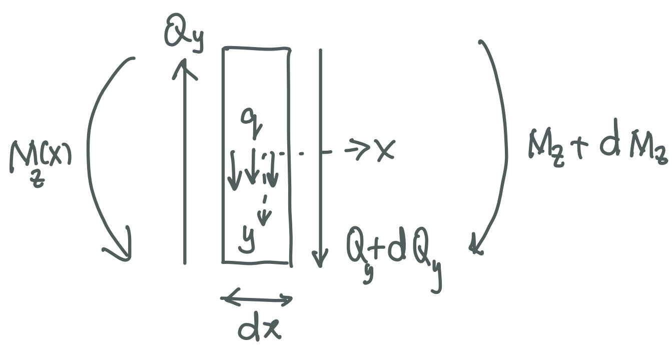


两者刚度不同, why?

①中的中性面无法承受剪切,平截面假设失效!

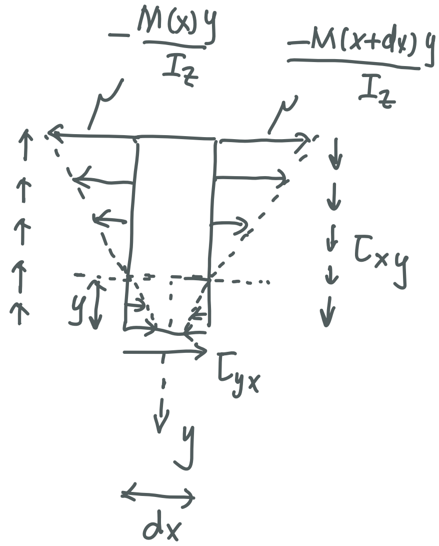
如何分析截面上的切应力?

我们目前已知: $\frac{dQ_y}{dx}=-q$, $\frac{dM_z}{dx}=-Q$, 以及 $\sigma_x=-\frac{M_z y}{I_z}$



进一步分析 Q_y, M_z 的根源

在y处切开, FBD

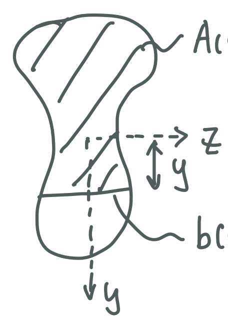


- $Q_y=\int_A \tau_{xy}\, dA$ (合力)

- $\Sigma F_x = -\int_{A^*} -\frac{M(x)y}{I_z}dA + \int_{A^*} -\frac{M(x+dx)y}{I_z}dA + \tau_{xy}(y)\cdot b(y)\cdot dx = 0$

$dA=dy\,dz$

A^*, $b(y)$是什么?



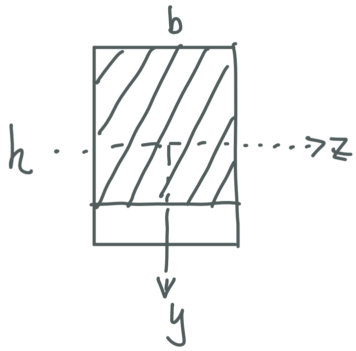
$$\rightarrow \tau_{xy}(x,y)\, b(y)\cdot dx = \int_{A^*}\frac{M(x+dx)-M(x)}{I_z}y'\,dA = \frac{dM(x)}{I_z}\underbrace{\int_{A^*} y\,dA}_{S_z^*}$$

S_z^*: A^*相对于z轴的静矩

$$\boxed{\begin{aligned}&\text{弯曲切应力：}\ \tau_{xy}(x,y) = -\frac{Q_y(x)\,S_z^*(y)}{I_z\, b(y)}\\ &\text{或}\ \tau_{xy} = -\frac{QS_z^*}{I_z b}\ \ (\text{简化记号})\end{aligned}}$$

显然 $\left.\begin{aligned}&S_z^*(y_{min}) = 0\\ &\underbrace{S_z^*(y_{max}) = 0}_{z\text{穿过形心}}\end{aligned}\right\}$ 物理含义?

矩形截面梁



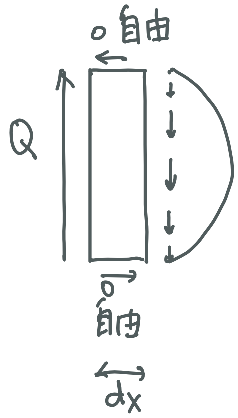
$b(y) \equiv b$

$$S_z^*(y) = \int_{-\frac{h}{2}}^{y} y\cdot b\,dy = \frac{b}{2}\left(y^2 - \frac{h^2}{4}\right)$$

$$\to \left|\frac{S_z^*}{b}\right|_{max} = \frac{1}{2}\left|y^2 - \frac{h^2}{4}\right|_{max} = \frac{h^2}{8}$$

$$\tau_{max} = \frac{Q}{\frac{1}{12}h^3 b}\cdot\frac{h^2}{8} = \frac{3}{2}\frac{Q}{hb}$$

$\tau_{average} = Q/A$?



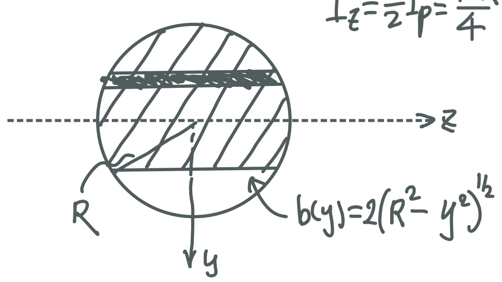
$$\tau_{xy}(y) = -\frac{12Q}{h^3 b^2}\cdot\frac{b}{2}\left(y^2 - \frac{h^2}{4}\right) = \frac{3}{2}\frac{Q}{hb}\left[1-\left(\frac{2y}{h}\right)^2\right],\ y\in\left[-\frac{h}{2},\frac{h}{2}\right]$$

$$\tau_{average} = \frac{1}{A}\int_A \tau_{xy}\,dA = \frac{1}{hb}\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{3}{2}\frac{Q}{hb}\left[1-\left(\frac{2y}{h}\right)^2\right]b\,dy$$

$$= \frac{3Q}{2h^2 b}\left(y - \frac{4}{3}\frac{y^3}{h^2}\right)\Bigg|_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{3Q}{2h^2 b}\left(h - \frac{4}{3}\cdot\frac{1}{4}h\right) = \frac{Q}{hb}$$ ✓

圆形截面梁

$I_z = \frac{1}{2}I_p = \frac{\pi R^4}{4}$



$$S_z^* = \int_{-R}^{y} y\cdot 2\sqrt{R^2-y^2}\,dy \xlongequal[du=2y\,dy]{u=y^2} \int_{R^2}^{y^2}(R^2-u)^{\frac{1}{2}}du$$

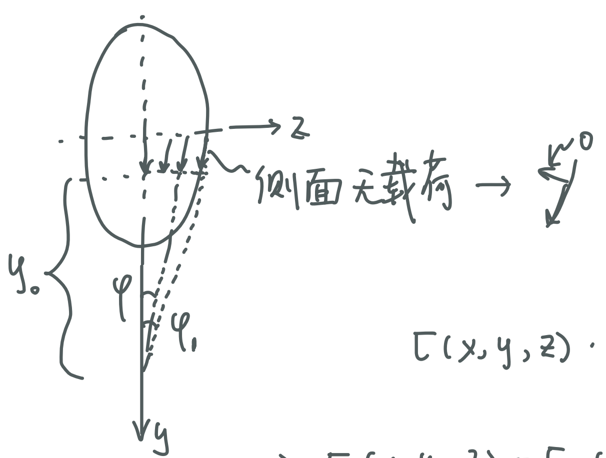
$$= \frac{-2}{3}(R^2-u)^{\frac{3}{2}}\Bigg|_{R^2}^{y^2} = -\frac{2}{3}(R^2-y^2)^{\frac{3}{2}}$$

$$\left|\frac{S_z^*}{b(y)}\right|_{max} = \frac{1}{3}(R^2-y^2)_{max} \xlongequal{\text{在 } y=0 \text{ 处}} \frac{1}{3}R^2$$

$$\rightarrow \tau_{max} = \frac{|Q|\frac{1}{3}R^2}{\frac{\pi}{4}R^4} = \frac{4}{3\pi}\frac{|Q|}{R^2} = \frac{4}{3}\frac{|Q|}{A} = \frac{4}{3}\tau_{average}$$

实际上，$\tau(x,y) = \dfrac{-Q(x)S_z^*(y)}{I_z(x)\,b(y)}$ 只是近似，仅对矩形截面较为适用。Why？思考以下两个例子。

①椭圆形截面：τ_{xy} 实际上是 τ 在 y 方向的投影



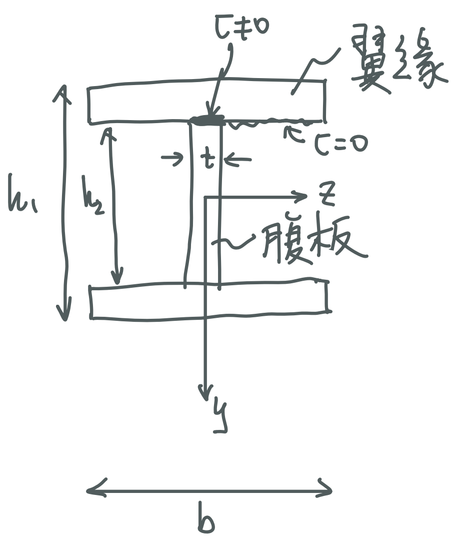
$$\tan\varphi_1 = \frac{b}{2y_0}$$

$$\tan\varphi = \frac{z}{y_0} = \tan\varphi_1 \frac{2z}{b}$$

$$\tau(x,y,z)\cdot\cos\varphi = \tau_{xy}$$

$$\rightarrow \tau(x,y,z) = \tau_{xy}(x,y)\,(1+\tan^2\varphi)^{1/2} = \tau_{xy}(x,y)\left[1+\left(\frac{2z}{b}\tan\varphi_1\right)^2\right]^{1/2}$$

② 工字梁：



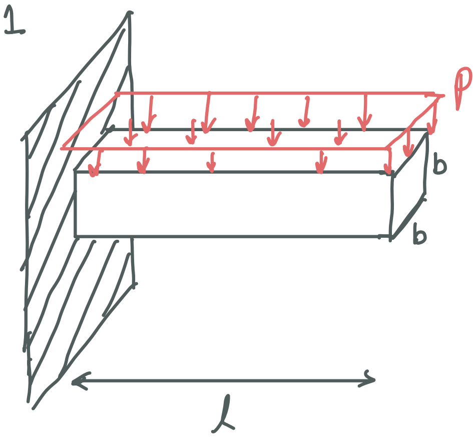
$$I_z = \frac{1}{12}h_1^3 b - \frac{1}{12}h_2^3(b-t)$$

$\tau = \dfrac{QS_z^*}{I_z b}$ 假设在 $b(y)$ 内均匀分布

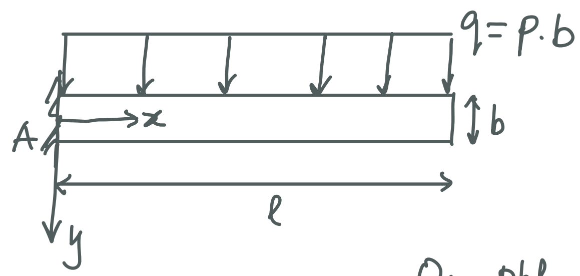
- $-\frac{h_1}{2} < y < \frac{1}{2}h_1$，合理.
- $|y| \geq \frac{h_2}{2}$ 时，不合理（在第七章中进一步讨论）.

一般情况下，$\sigma_{yy}\neq 0$，$\tau_{xy}\neq 0$，但我们只考虑了 σ_{xx} 引起的变形 ($M_z = EIK$). why?

例1

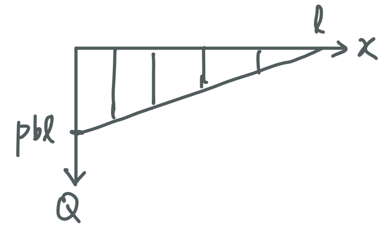


→

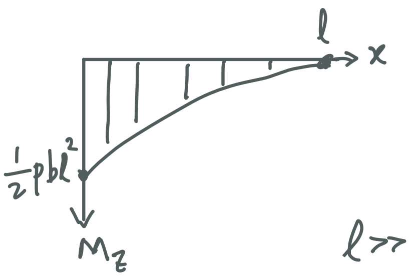


$Q_A = pbl$

$M_A = \frac{1}{2}pbl^2$



$$|\sigma_x|_{max} = \left|\frac{M_z\cdot y}{I_z}\right|_{max} = \frac{\frac{1}{2}pbl^2\cdot\frac{b}{2}}{\frac{1}{12}b^4} = 3p\left(\frac{l}{b}\right)^2$$



$$|\tau_{xy}|_{max} = \left|\frac{Q\,S_z^*}{I_z}\right|_{max} = \frac{3}{2}\cdot\frac{pbl}{b^2} = \frac{3}{2}p\left(\frac{l}{b}\right)$$

$$l \gg b \longrightarrow \boxed{|\sigma_x| \sim p\left(\frac{l}{b}\right)^2 \gg |\tau_{xy}| \sim p\left(\frac{l}{b}\right) \gg |\sigma_y| \sim p}$$

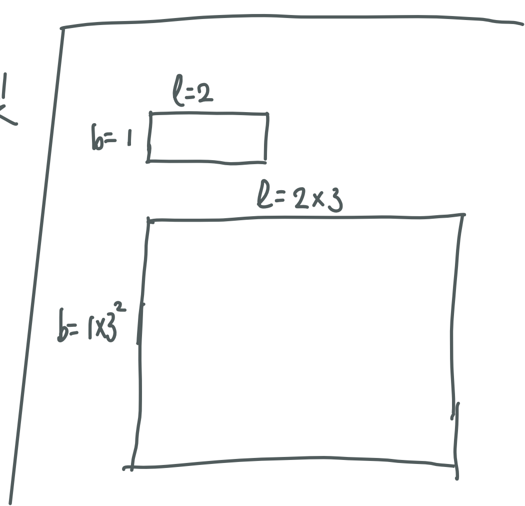
重新考虑伽里略的地狱厚度问题

• 外力：ρg (单位体积) → $\underbrace{\rho g b}_{p}$ (单位面积) → $\rho g b^2$ (单位长度)

• 当 $|\sigma_x|_{max} = \sigma_0$ 时，梁在重力下在 $x=0$，$y=-\frac{h}{2}$ 处断裂

$$\rightarrow \quad \sigma_0 = 3\cdot\rho g b\cdot\left(\frac{l}{b}\right)^2 = 3\rho g\frac{l^2}{b}$$

∴ 长度增加 α 倍，厚度需增加 α^2 倍 e.g.



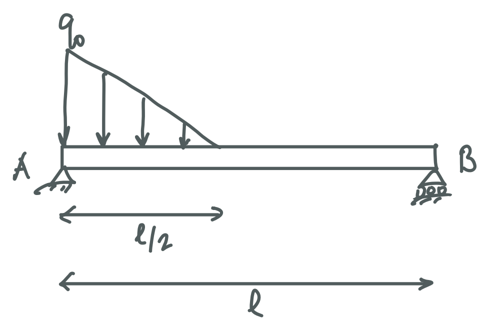
若考虑切应带来的变形，相应的弹性变形能则为

$$U = \underset{\substack{\uparrow \\ \text{正应}}}{U_M} + \underset{\substack{\uparrow \\ \text{切应}}}{U_Q} \,,$$

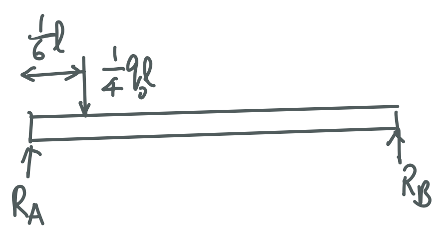
(两者可解耦 $\sigma \leftrightarrow \tau \updownarrow$)

$$\frac{U_M}{U_Q} \sim \left(\frac{l}{h}\right)^2 \gg 1 \leftarrow \begin{cases} U_M = \int_0^l \frac{M_z^2}{2EI_z} dl. \ (\text{上节课内容}) \sim \frac{(ql^2)^2 \cdot l}{E h^3 b} \sim \underbrace{\frac{q^2 l}{E}}_{[J]} \frac{l^4}{h^3 b} \\ U_Q = \int_0^l \underbrace{\left(\frac{Q}{dA}\right)^2}_{\tau_{averge}} \cdot \frac{1}{2G} dA\, dl \sim \int_0^l \frac{Q^2}{2GA} dl \sim \frac{q^2 l^2}{E h b} l \sim \frac{q^2 l}{E} \frac{l^2}{hb} \end{cases}$$

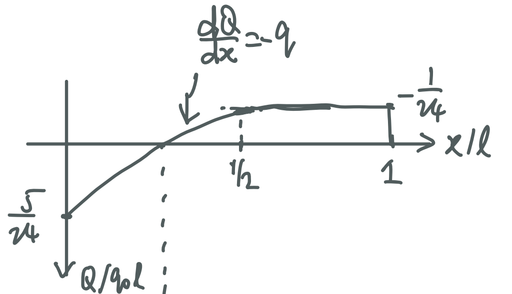
例2.



σ_{max}，τ_{max} 以及其位置

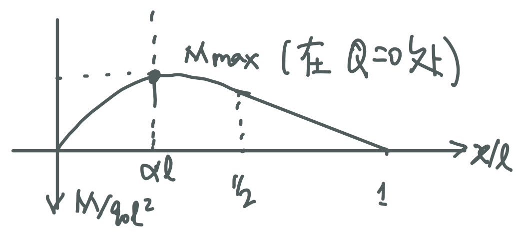


$$R_A + R_B = \frac{1}{4} q_0 l \qquad R_A = \frac{5}{24} q_0 l$$
$$R_B \cdot l = \frac{1}{24} q_0 l^2 \quad \rightarrow \quad R_B = \frac{1}{24} q_0 l$$



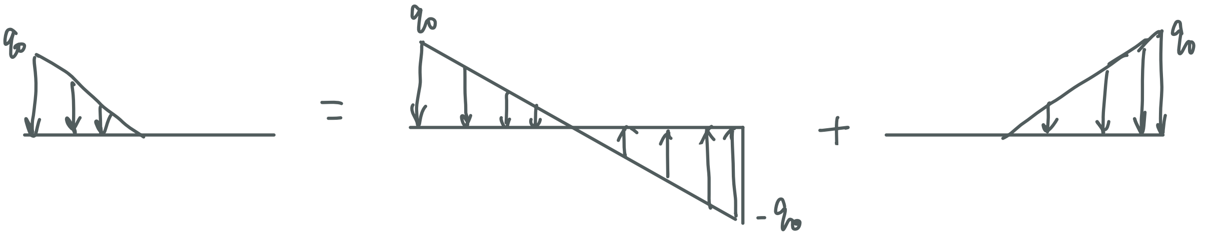
$|Q|_{max} = \frac{5}{24} q_0 l$，发生在 $x=0$ 处

$$\tau_{max} = \left(\frac{QS}{I_z b}\right)_{max} = \frac{5}{24} q_0 l \times \frac{1}{hb} \times \frac{3}{2} = \frac{5}{16} \frac{q_0 l}{hb}$$



需要求解 M&Q.

方法①



$$q(x)=q_0-2q_0\frac{x}{l}+2\frac{q_0}{l}\varphi_1\left(x-\frac{1}{2}l\right)$$

$$Q(x)=-q_0x+\frac{q_0x^2}{l}-\frac{2q_0}{l}\varphi_2\left(x-\frac{1}{2}l\right)+\frac{5}{24}q_0l$$

[Check: $Q(l)=q_0l\left(-1+1-\frac{1}{4}+\frac{5}{24}\right)=-\frac{1}{24}q_0l$ ✓]

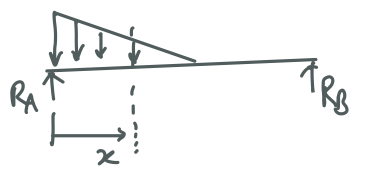
$$\rightarrow \frac{5}{24}-\alpha+\alpha^2=0 \rightarrow \alpha\simeq 0.296$$

$$M(x)=+\frac{1}{2}q_0x^2-\frac{1}{3}\frac{q_0x^3}{l}+\frac{2q_0}{l}\varphi_3\left(x-\frac{1}{2}l\right)-\frac{5}{24}q_0lx+\cancel{M_z(0)}^{\,0}$$

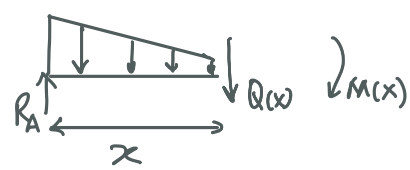
[Check: $\frac{M(l)}{q_0l^2}=\frac{1}{2}-\frac{1}{3}+\frac{2}{3!}\frac{1}{8}-\frac{5}{24}=0$ ✓]

$$\rightarrow M_{max}=M(x=\alpha l)=\left|q_0l^2\left(+\frac{1}{2}\alpha^2-\frac{1}{3}\alpha^3-\frac{5}{24}\alpha\right)\right|\simeq 0.265\,q_0l^2$$

方法②

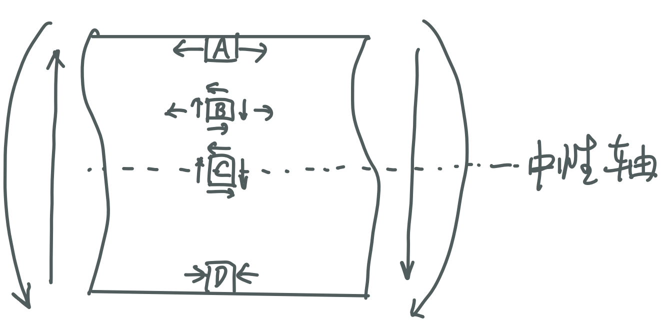


$x<\frac{l}{2}$时：



§4.4. 梁的强度条件和梁的合理设计

强度条件



A, D处：无切应力，$|\sigma_x|$最大

$$|\sigma_x|_{max}\le[\sigma]$$

(类似于拉压问题，但此处$[\sigma]$更大一些)

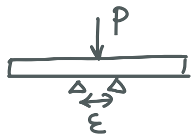
C处：无正应力，$|\sigma_{xy}|$最大，$|\tau_{xy}| \le [\tau]$（类似于纯剪切问题）

B处：复杂应力状态，强度理论进行计算（后续内容）

一般情况下，$|\sigma_x|_{max} >> |\tau_{xy}|_{max}$，$[\sigma] \sim [\tau]$，A、D处最为关键，无需考虑剪切.

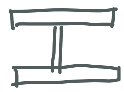
有以下几个特殊情况

- 最大M较小，最大Q较大时.



$M \sim P \cdot \varepsilon$，$Q \sim P$.

- 组合截面，如工字梁腹板较薄时

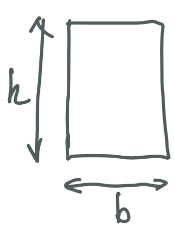


- 木梁等顺纹方向抗剪强度较差时

优化措施

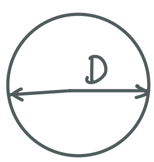
$$|\sigma_x|_{max} = \frac{|M_z|_{max}\,|y|_{max}}{I_z} = \frac{|M_z|_{max}}{W}, \quad W = \frac{I_z}{|y|_{max}}$$ —抗弯截面系数.

- 增大 W/A（相同材料面积下，提高W）



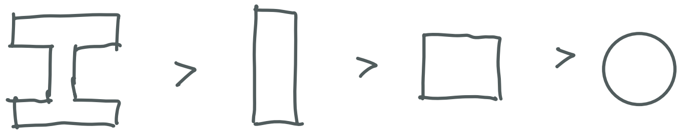
$$W = \frac{1}{12}h^3 b \Big/ \frac{1}{2}h = \frac{1}{6}Ah \simeq 0.167\,Ah$$

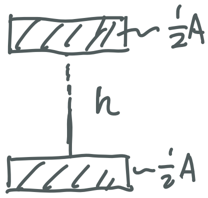
相同面积下，h越大越好.



$$W = \frac{\pi D^4}{64} \Big/ \frac{1}{2}D = \frac{\pi}{32}D^4 = \frac{1}{8}AD \simeq 0.125\,Ah < W_{square}$$

相同面积下，材料离中性轴越远越好.



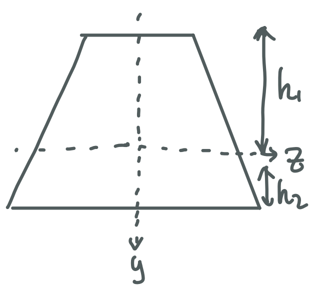
思考: 

$I_z \approx 2\times\left(\frac{h}{2}\right)^2\times\frac{A}{2} = \frac{1}{4}Ah^2$ (平移轴定理)

$W = 0.5Ah$

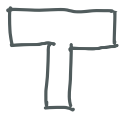
• 拉伸和压缩应力的强度储备相同

当 $[\sigma]_t \neq [\sigma]_c$ 时，使得 $\frac{|\sigma^+_{max}|}{|\sigma^-_{max}|} = \frac{[\sigma]_t}{[\sigma]_c}$

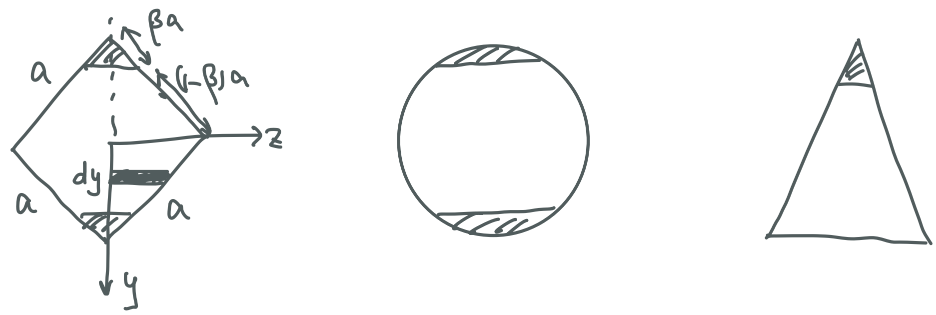


上拉下压时. $\left|\frac{M_z h_1}{I_z}\right|_{max} \Big/ \left|\frac{M_z h_2}{I_z}\right|_{max} = \frac{h_1}{h_2} = \frac{[\sigma]_t}{[\sigma]_c}$

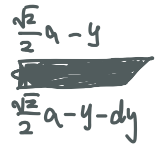
反之，$\frac{h_1}{h_2} = \frac{[\sigma]_c}{[\sigma]_t}$

其它形状 ，道理相同.

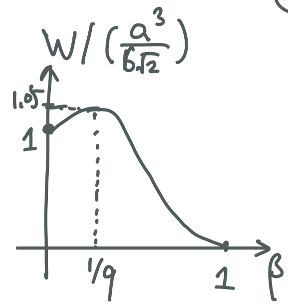
• 削去上下端处的小部分面积 ($W = \frac{I_z}{y_{max}}$，当 y_{max} 比 I_z 减小的"更快"时)



$I_z(\beta) = 2\times\int_0^{\frac{\sqrt{2}}{2}(1-\beta)a} y^2\,dA$，$dA = dy\times2\times\left(\frac{\sqrt{2}}{2}a-y\right)$



$$= \frac{a^4}{12}(1-\beta)^3(1+3\beta) = \frac{a^4}{12}\left(1-6\beta^2+O(\beta^3)\right) \quad \text{for } \beta \ll 1$$

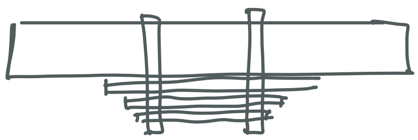


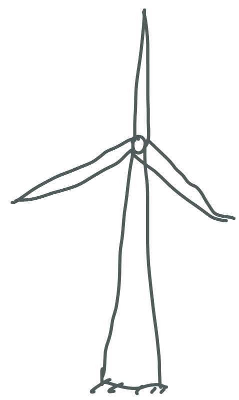
$y_{max}(\beta) = \frac{\sqrt{2}}{2}(1-\beta)a$ （当 $\beta \ll 1$，减小的比 I_z 更快）

$$W = I_z / y_{max} = \frac{a^3}{6\sqrt{2}}(1-\beta)^2(1+3\beta) = \frac{a^3}{6\sqrt{2}}\left(1+\beta-5\beta^2+O(\beta^3)\right) \quad \text{Good!}$$

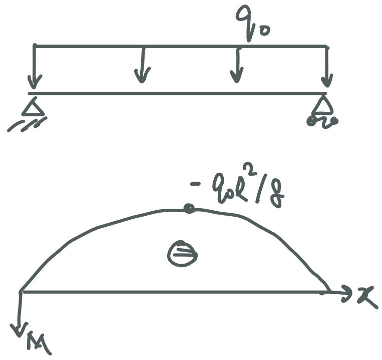
- 变截面梁：在弯矩大的位置，使用更大的 W（更经济）

$$\sigma_{max} = \frac{M(x)}{W(x)} = [\sigma]$$

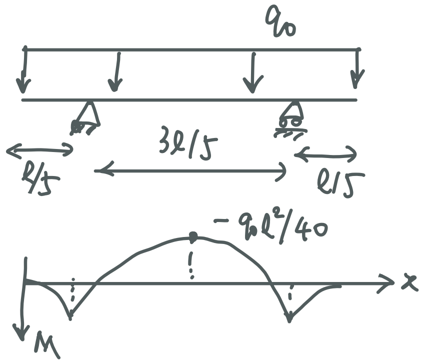


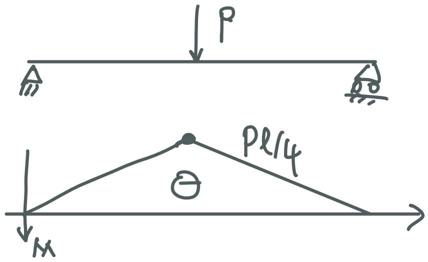


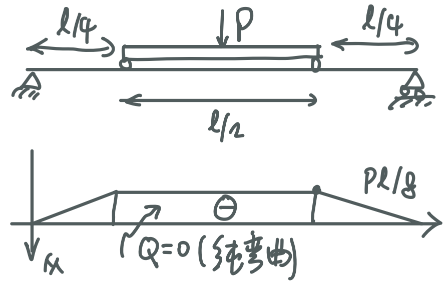
- 合理安排约束&加载方式



⟹



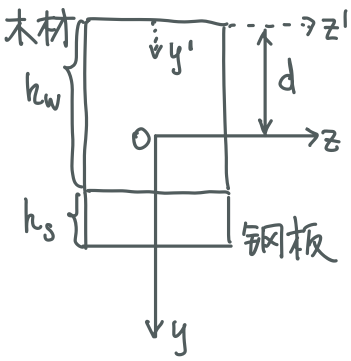




§5.5. 两种材料的组合梁

最后，简单推导由两个梁"紧密"连接组合梁中的物理关系

$\underbrace{\text{紧密}}_{\text{界面无滑动}}$



(中性轴在哪?)

平截面假设 (E_s, E_w 不能相差太大)：

$$\sigma_w = -\frac{1}{R} E_w y$$

$$\sigma_s = -\frac{1}{R} E_s y$$

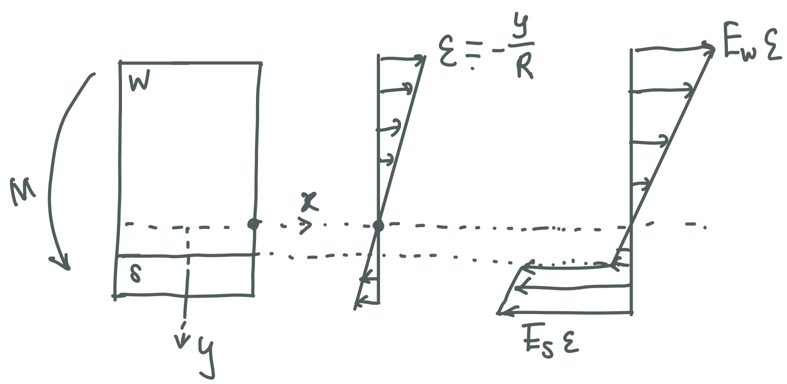
$\underline{\Sigma F_x = 0}$：
$$\int_{A_w} E_w y \, dA + \int_{A_s} E_s y \, dA = 0$$

$$y = y' - d \to \int_{A_w} E_w (y'-d) \, dA + \int_{A_s} E_s (y'-d) \, dA = 0$$

$$\to d = \frac{E_w \int_{A_w} y' dA + E_s \int_{A_s} y' dA}{E_w A_w + E_s A_s}$$

Check：当 $E_w = E_s$ 时，$d = \int_{A_w + A_s} y' \, dA \,/\, (A_w + A_s) = \frac{1}{2}(h_w + h_s)$ ✓

$\underline{\Sigma M_z = 0}$：
$$\int_{A_w} E_w \cdot \left(-\frac{y}{R}\right)(-y) \, dA + \int_{A_s} E_s \left(-\frac{y}{R}\right)(-y) \, dA = M$$



$$\to M = (E_w I_w + E_s I_s) \cdot \frac{1}{R}$$

$$\to \sigma_w = \frac{-E_w M y}{E_w I_w + E_s I_s}$$

$$\sigma_s = \frac{-E_s M y}{E_w I_w + E_s I_s}$$