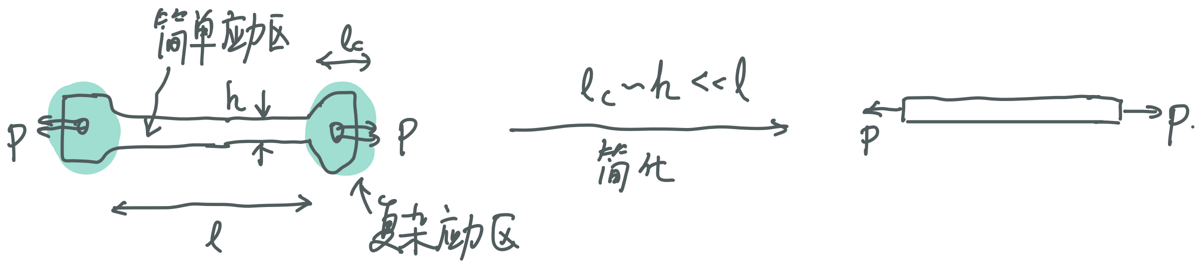
# §2.1 直杆的拉伸和压缩

我们着先关注直杆(系)在外力作用下产生的变形和内力. 为此,采用以下假设(约定):

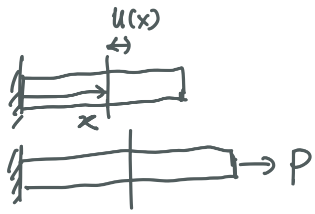
· 圣维南原理

端部作用力的形式通常较为复杂,但该复杂性只体现在一个可忽略的范围.



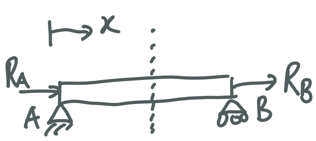

· 平截面假设

原有的横截面在变形后仍为平面,且与轴线垂直.

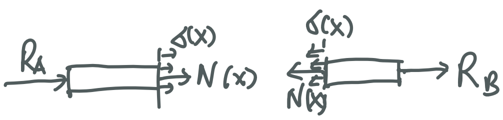



$u=u(x)$,不依赖 $y, z$.
(三维问题 → 一维化)

· 符号约定

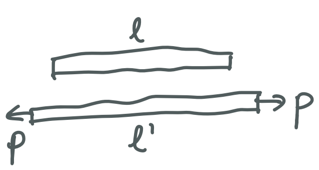



· 约束通常与坐标方向一致



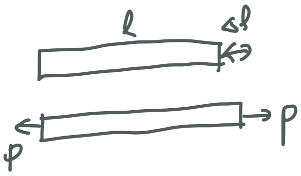

· 内力/应力与截面法向方向一致, $\sigma>0$ (拉应力)
$\sigma<0$ (压应力)

· 小变形假设 & Hooke's law




$\varepsilon = \frac{l'-l}{l} \ll 1$ 静力平衡可在变形前建立.

- 胡克定律

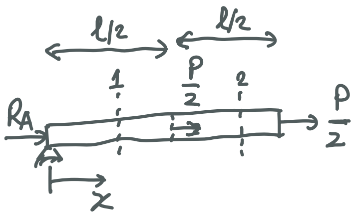



$$\varepsilon=\frac{\Delta l}{l}=\frac{\sigma}{E}=\frac{P}{EA}\Rightarrow P=\underbrace{\frac{EA}{l}}\Delta l$$

k - 拉压刚度.

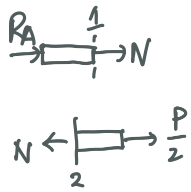
EA - 截面刚度系数.

例1 杆的内力与变形




①平衡关系

$\Sigma F_x=0 \to R_A=-P$ （静定问题）




$$\Rightarrow N=\begin{cases}P, & 0\le x<l/2\\ \frac{P}{2}, & \frac{l}{2}<x\le l\end{cases} \quad \& \quad \sigma=\begin{cases}P/A, & 0\le x<\frac{l}{2}\\ \frac{P}{2A}, & \frac{l}{2}<x\le l\end{cases}$$

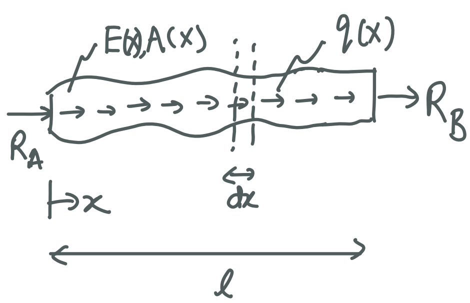
② 胡克定律

$$\varepsilon=\begin{cases}\frac{P}{EA}, & 0\le x<\frac{l}{2}\\ \frac{P}{2EA}, & \frac{l}{2}<x\le l\end{cases} \to \begin{array}{l}\Delta l_1=\varepsilon_1\cdot l_1=\frac{P}{EA}\cdot\frac{l}{2}\\ \Delta l_2=\frac{P}{2EA}\cdot\frac{l}{2}\end{array} \to \Delta l=\Delta l_1+\Delta l_2=\frac{3}{4}\frac{Pl}{EA}$$

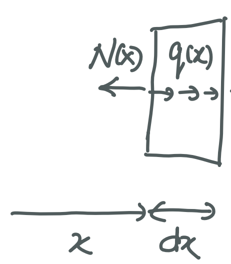
对于这个简单的问题，我们可以直接根据"刚度"得出 $\Delta l_1=\frac{N_1}{k_1}=\frac{P}{EA}\frac{l}{2}$, $\Delta l_2=\frac{N_2}{k_2}=\frac{Pl}{4EA}$

但是对于更为一般形式的系统（N非常数），怎么处理？— 微元的FBD

## §2.2 非均匀变形杆




①平衡方程



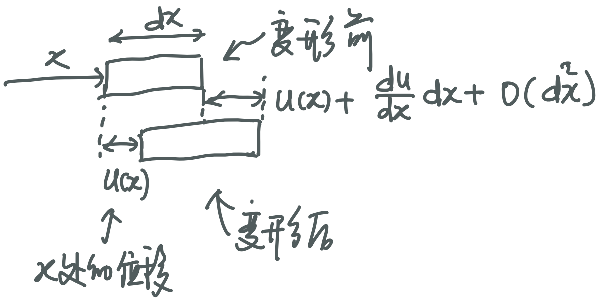

$$N(x)+\frac{dN}{dx}dx+O(dx^2)$$

也可取左右半部得到积分方程

$$\cancel{N(x)} = q(x)\,dx + \cancel{N(x)} + \frac{dN}{dx}dx + \cancel{O(dx^2)}^{\to 0} \to \boxed{\frac{dN}{dx} + q(x) = 0}$$

可直接积分解答: $N(x) = \underbrace{N_0}_{-R_A} - \int_0^x q(x)\,dx$ ，以及 $\sigma(x) = \frac{N(x)}{A(x)}$

② 非均匀应变




$l = dx$

$l' = dx + \cancel{u(x)} + \frac{du}{dx}dx - \cancel{u(x)}$

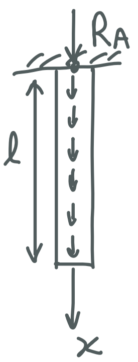
$$\boxed{\varepsilon(x) = \frac{l' - l}{l} = \frac{du}{dx}}$$

③ 胡克定律

$$\varepsilon(x) = \frac{N(x)}{E(x)A(x)} = \frac{du}{dx} \quad \text{(物理方程)}$$

伸长量 $u(l) - u(0) = \int_0^l \frac{du}{dx}dx = \int_0^l \frac{N(x)}{E(x)A(x)}dx$

例2. 自重引起的杆的应力和变形（矿井升降机吊索 or 石油钻井机钻杆）




$q(x) = \rho g A$ （单位长度）　　$\rho g A l = W$ ← 自重

$\to q(x) = \frac{W}{l}$

① 平衡方程

$\Sigma F_x = 0 \to R_A = -W$

$\frac{dN}{dx} + q(x) = 0 \to N = W - \int_0^x \frac{W}{l}dx = \frac{W}{l}(l - x)$

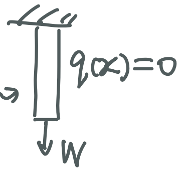
$\sigma = \frac{N}{A} = \rho g(l - x)$

② 物理方程

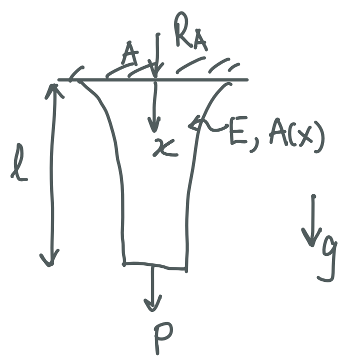
$$\varepsilon(x)=\frac{du}{dx}=\frac{N}{EA}=\frac{W}{EAl}(l-x)$$

$$\underbrace{u(l)-\cancel{u(0)}^{\,0}}_{\Delta l}=\int_0^l \varepsilon\,dx=\frac{Wl}{2EA}$$

相差于 $\frac{1}{2}$




例3. 如何设计不同截面上应力值相同的杆？（使"材料强度"效益最大）




① 平衡方程

$$q(x)=\rho g A(x)$$

$$R_A=-P-\int_0^l \rho g A\,dx$$

$$N(x)=P+\int_0^l \rho g A\,dx-\int_0^x \rho g A\,dx=A\cdot\sigma_0$$ ← 常数

两侧对 $x$ 求导 → $\frac{dA}{dx}\cdot\sigma_0+\rho g A=0$ → $A=A(0)e^{-\frac{\rho g}{\sigma_0}x}$

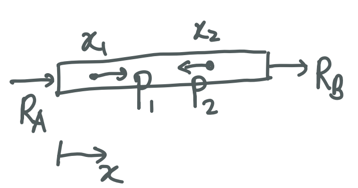
此时 $R_A=-P-A(0)\sigma_0\left(1-e^{-\frac{\rho g}{\sigma_0}l}\right)$

$$\sigma_0=\frac{-R_A}{A(0)}=\frac{P}{A(0)}+\sigma_0\left(1-e^{-\frac{\rho g}{\sigma_0}l}\right)$$

可以让 $\sigma_0=\sigma_s$，$P$ 给定，得出 $A_c(0)$. 设计时应 $A(0)>A_c(0)$

屈服应力

例4. 集中力



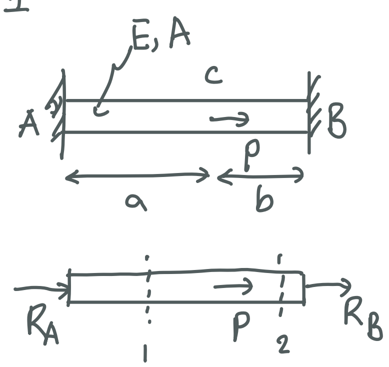

集中力也可以做为分布力 例如 $q(x)=P_1\delta(x-x_1)-P_2\delta(x-x_2)$

但对该问题并未带来实际上的便利. 我们将会在后续再讨论

# §2.3 静不定问题

上面的例子中，我们仅用平衡方程便可求解支反力，这类问题为静定问题。但在很多问题中，仅用平衡方程无法确定支反力&内力—静不定(statically indeterminate)问题。此时，除了平衡、物理方程外，我们还需几何(协调)方程)。

例1



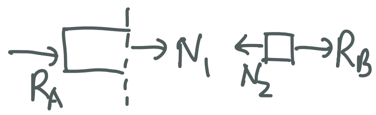

① 平衡方程

$$R_A+R_B+P=0$$

$$N_1=-R_A$$

$$N_2=R_B$$

4未知量，3方程




② 物理方程

$$\Delta l_1=\frac{N_1 a}{EA}, \quad \Delta l_2=\frac{N_2 b}{EA}$$

并未解决问题(2个新量)

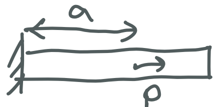
③ 几何方程

$$\Delta l_{AB}=\Delta l_1+\Delta l_2=-\frac{R_A a}{EA}+\frac{R_B b}{EA}=0 \rightarrow R_A=\frac{b}{a}R_B$$

$$\rightarrow R_A=\frac{-b}{a+b}P, \quad R_B=\frac{-a}{a+b}P$$

例2. 叠加法(superposition)求解上述问题

分解为两个线性问题的叠加

①



$$R_A^{①}=-P, \quad \Delta l^{①}=\frac{Pa}{EA}$$

+思想实验

②



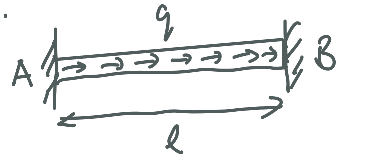
$$R_A^{②}=-R_B^{②}, \quad \Delta l^{②}=\frac{R_B(a+b)}{EA}$$

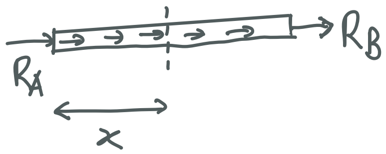
$$\Delta l = \Delta l^{①} + \Delta l^{②} = 0 \rightarrow R_B^{②} = -\frac{a}{a+b}P$$

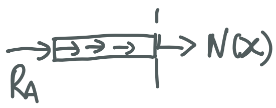
$$R_A = R_A^{①} + R_A^{②} = -P - R_B^{②} = -\frac{b}{a+b}P, \quad R_B = \cancel{R_B^{①}}^{0} + R_B^{②} = -\frac{a}{a+b}P.$$

同样 $\Delta l_1 = \Delta l_1^{①} + \Delta l_1^{②}$， $\Delta l_2 = \Delta l_2^{①} + \Delta l_2^{②}$

例 3.




① 平衡方程

$$R_A + R_B + qL = 0$$

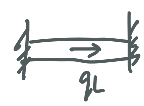
$$N(x) = -R_A - qx \quad (\text{也可由 } \frac{dN}{dx} + q = 0 \text{ 得出})$$

② 物理方程

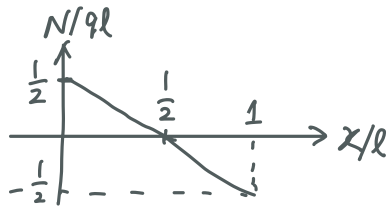
$$\frac{du}{dx} = \varepsilon = \frac{N(x)}{EA} = -\frac{R_A}{EA} - \frac{q}{EA}x$$

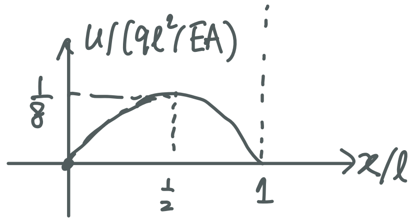
③ 几何方程

$$\cancel{u(l)}^{0} - \cancel{u(0)}^{0} = \int_0^l \left(-\frac{R_A}{EA} - \frac{q}{EA}x\right)dx = 0$$

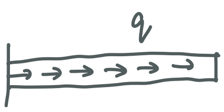
$$\rightarrow R_A = -\frac{ql}{2}, \quad R_B = -\frac{ql}{2} \quad ($$  $$)$$

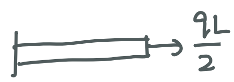

$$\rightarrow N(x) = ql\left(\frac{1}{2} - \frac{x}{l}\right), \quad u(x) = \frac{ql^2}{EA}\left(\frac{1}{2}\frac{x}{l} - \frac{1}{2}\frac{x^2}{l^2}\right)$$



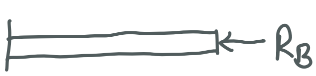




也可采用叠加法：去除左侧约束，先施加 q ①，再施加 $R_B$ ②

① 


③ 

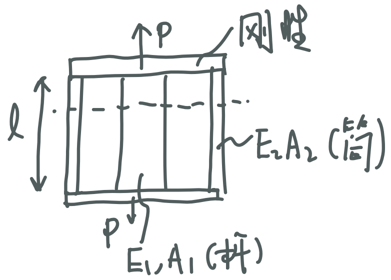

$$\Delta l^{①} = \Delta l^{③} = \frac{ql}{2EA}$$

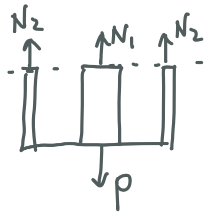
② 


$$\Delta l^{②} = -\frac{R_B l}{EA}$$

$$\Delta l = \Delta l^{①} + \Delta l^{②} = 0 \rightarrow R_B = \frac{ql}{2}$$

例4. 组合杆




平衡方程

$$N_1 + N_2 = P$$

物理方程

$$\Delta l_1 = \frac{N_1 l}{E_1A_1}, \quad \Delta l_2 = \frac{N_2 l}{E_2A_2}$$

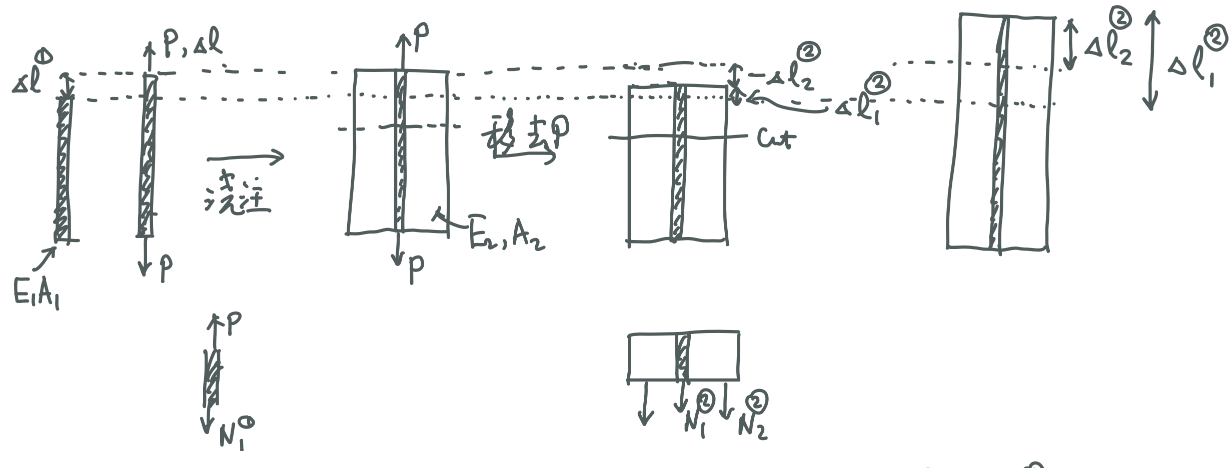
几何方程

$$\Delta l_1 = \Delta l_2$$

$$\rightarrow N_1 = \frac{E_1A_1}{E_1A_1 + E_2A_2}P, \quad N_2 = \frac{E_2A_2}{E_1A_1+E_2A_2}P$$

$$\rightarrow \Delta l = \frac{Pl}{E_1A_1+E_2A_2} \quad \left(k = \frac{E_1A_1}{l} + \frac{E_2A_2}{l} = k_1 + k_2\right)$$

例5. 预应力混凝土




平衡方程: $N_1^{①} = P$

$$N_1^{②} + N_2^{②} = 0 \rightarrow N_1^{②} = -N_2^{②} = N$$

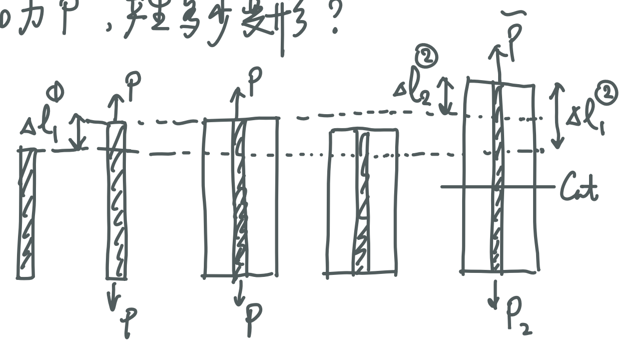
物理方程: $\Delta l_1^{①} = \frac{Pl}{E_1A_1}$

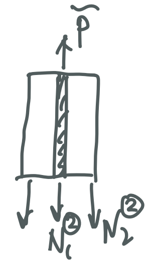
$$\Delta l_1^{②} = \frac{Nl}{E_1A_1}, \quad \Delta l_2^{②} = \frac{-Nl}{E_2A_2}$$

几何方程: $\Delta l_1^{①} = \Delta l_1^{②} + (-\Delta l_2^{②})$ (不依赖如何画FBD)

$$\rightarrow N = \frac{E_2A_2}{E_1A_1+E_2A_2}P, \quad \sigma_1 = \frac{E_2A_2}{(E_1A_1+E_2A_2)A_1}P, \quad \sigma_2 = \frac{-E_2}{E_1A_1+E_2A_2}P. \text{(受压)}$$

再施加力 $\tilde{P}$，产生多少变形？




$$\Delta l_1^{①}=\frac{Pl}{E_1A_1}$$

$$\begin{cases} N_1^{②}+N_2^{②}=\tilde{P} & \text{平衡} \\ \Delta l_1^{②}=\dfrac{N_1^{②}l}{E_1A_1},\quad \Delta l_2^{②}=\dfrac{N_2^{②}l}{E_2A_2} & \text{物理} \\ \Delta l_1^{①}+\Delta l_2^{②}=\Delta l_1^{②} & \text{几何} \end{cases}$$

$$\rightarrow N_1^{②}=\frac{E_2A_2P+E_1A_1\tilde{P}}{E_1A_1+E_2A_2},\quad N_2^{②}=\frac{E_2A_2}{E_1A_1+E_2A_2}(\tilde{P}-P)\qquad \tilde{P}=P,\ N_1^{②}=P,\ N_2^{②}=0\ \checkmark$$

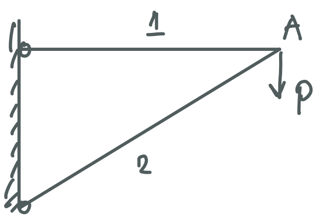
相对变形 $\Delta l(\tilde{P})-\Delta l(\tilde{P}=0)=\dfrac{N_1^{②}(\tilde{P})\,l}{E_1A_1}-\dfrac{N_1^{②}(\tilde{P}=0)\,l}{E_1A_1}$ （也可采用 $N_2^{②}$）

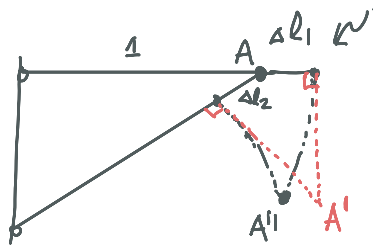
$$=\frac{\tilde{P}l}{E_1A_1+E_2A_2}=\frac{\tilde{P}}{k_1+k_2}$$

∴预应力可以使得混凝土在 $\tilde{P}<P$ 内保持压缩状态，但并不会改变整体刚度！

## §2.4 简单桁架

桁架结构中，各杆只受轴力，且在节点处可自由转动。我们已经学习了通过FBD求解第i个杆中的轴力 $N_i$，根据物理方程可知其伸长量为 $\Delta l_i=\dfrac{N_i l_i}{E_iA_i}$，如何求节点的变形？—维利奥特(Williot)图



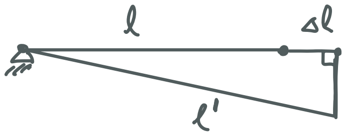




← 夸张错觉，真实 $\frac{\Delta l}{l}\sim10^{-3}$ (why?)

几何简单，但计算繁琐。

利用切线代替圆弧！

误差估计:



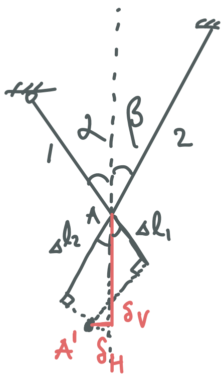

$u \sim \Delta l \ll l$ 忽略转动的弧长.

→0 高阶量.

$$l' = \sqrt{(l+\Delta l)^2+u^2} = (l+\Delta l)\sqrt{1+\left(\frac{u}{l+\Delta l}\right)^2} = (l+\Delta l)\left[1+\frac{1}{2}\cancel{\left(\frac{u}{l+\Delta l}\right)^2}+\cdots\right] \simeq l+\Delta l$$

→小变形下，切线代替圆弧的误差为 $O\left(\frac{\Delta l^2}{l^2}\right)$，更关注 $\frac{\Delta l}{l}\sim 10^{-3}$ 量级.

例1

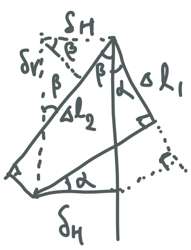



$\delta_H(\downarrow)$，$\delta_V(\leftarrow)$ 分别为水平和铅直方向位移.

几何方程:

$$\Delta l_2 = \delta_H \sin\beta + \delta_V \cos\beta$$

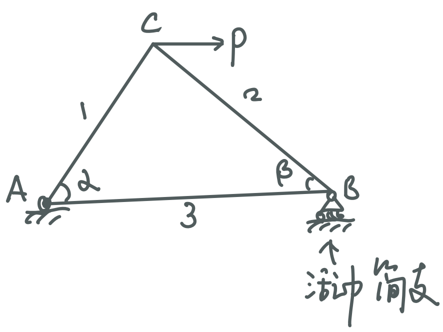
$$\Delta l_1 = \delta_V \cos\alpha - \delta_H \sin\alpha$$

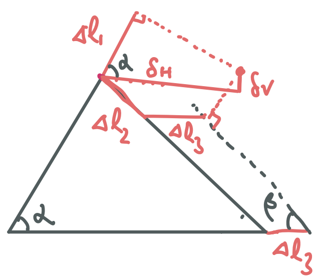


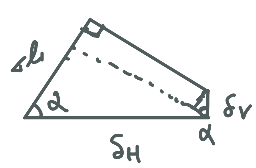

$$\rightarrow \delta_H = \frac{\Delta l_2\cos\alpha - \Delta l_1\cos\beta}{\sin(\alpha+\beta)}, \quad \delta_V = \frac{\Delta l_1\sin\beta + \Delta l_2\sin\alpha}{\sin(\alpha+\beta)}$$

Check: $l_1 = l_2$, $\alpha=\beta$, $E_1A_1 = E_2A_2$, $\Delta l_1 = \Delta l_2$, $\delta_H = 0$ ✓ (对称条件).

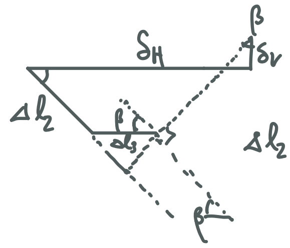
例2.



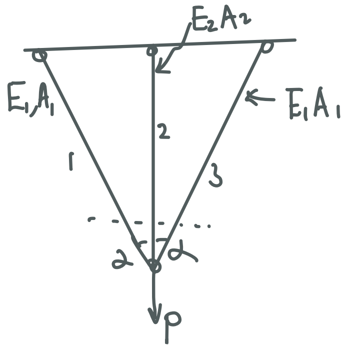




$$\Delta l_1 = \delta_H\cos\alpha + \delta_V\sin\alpha$$

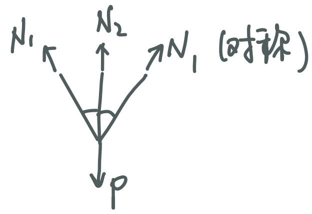



$$\Delta l_2 + \Delta l_3\cos\beta = \delta_H\cos\beta - \delta_V\sin\beta$$

## 例3. 静不定桁架

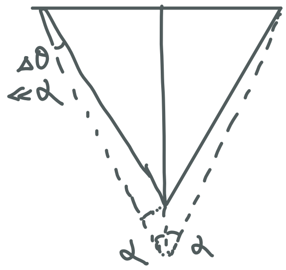



平衡方程




$$2N_1\cos\alpha + N_2 = P$$

物理方程 $\Delta l_1 = \Delta l_3 = \dfrac{N_1 l}{E_1A_1\cos\alpha}$ , $\Delta l_2 = \dfrac{N_2 l}{E_2A_2}$



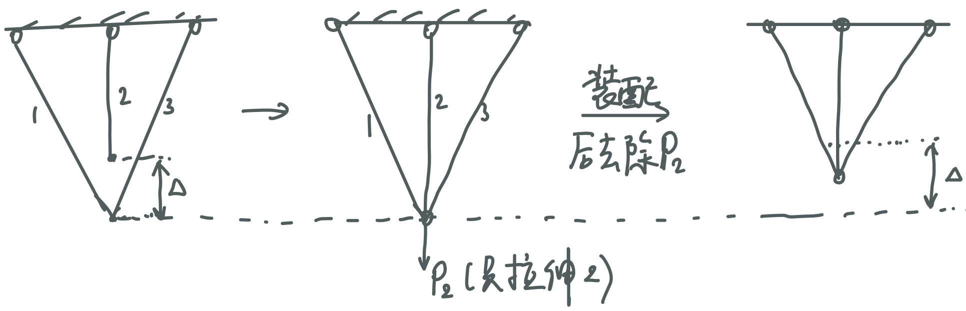

几何方程 $\Delta l_1 = \Delta l_2 \cos\alpha$

$$\Rightarrow \frac{N_1}{E_1A_1\cos\alpha} = \frac{N_2}{E_2A_2}\cos\alpha$$

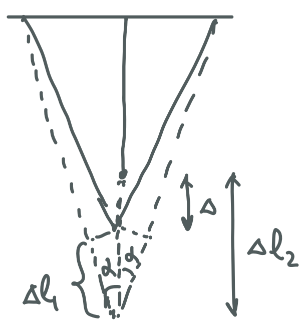
$$\to N_1 = N_3 = \frac{E_1A_1\cos^2\alpha}{E_2A_2+2E_1A_1\cos^3\alpha}P,\quad N_2 = \frac{E_2A_2}{E_2A_2+2E_1A_1\cos^3\alpha}P$$

$\delta_H = 0$ , $\delta_V = Pl/(E_2A_2+2E_1A_1\cos^3\alpha)$ (what if $\alpha\to\frac{\pi}{2}$?)

## 例4: 装配应力




几何方程

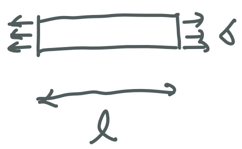



$$(\Delta l_2 - \Delta)\cos\alpha = \Delta l_1$$

思考：再施加P到装配后的结构，产生了多少位移？

# §2.5 弹性变形能.

外力所作功会以弹性变形能储存在物体内。对均匀应力的直杆，我们可以轻易得出

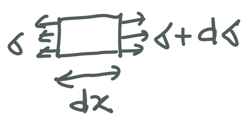



$$U=\int_0^{\Delta l} P\, d(\Delta l)$$

$$=\int_0^{\varepsilon} \sigma A\, d(\varepsilon\cdot l) \xrightarrow[\sigma=E\varepsilon]{\text{胡克材料}} U_0=\frac{1}{2}\frac{\sigma^2}{E}=\frac{1}{2}E\varepsilon^2=\frac{1}{2}\sigma\varepsilon.$$

$$=\underset{Al}{V}\underbrace{\int_0^{\varepsilon}\sigma\, d\varepsilon}_{U_0\ \text{应变比能}}$$

对于非均匀应力直杆，可以考虑微元内为均匀应力:




$$U=\int_V dV\int_0^{\varepsilon}\sigma\, d\varepsilon = A\int_0^l dx\, U_0(x) = \frac{1}{2}\int_0^l EA\varepsilon^2 dx$$

$$\boxed{=\frac{1}{2}\int_0^l \frac{N^2(x)}{EA}dx}$$

当N为常数时 $U=\frac{N^2 l}{2EA}$.

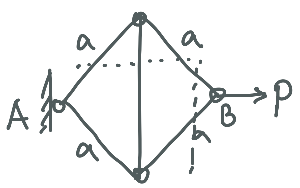
可以把弹性体看做是一个保守的力学系统，i.e., $\underbrace{\text{外力功}}_{W}$全部转化为$\underbrace{\text{动能}}_{T}$和$\underbrace{\text{弹性变形能}}_{U}$

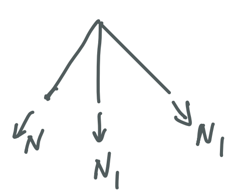
$$W=\cancel{T}^{\,0}+U$$

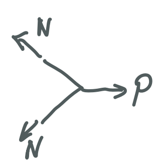
动能在"小"加载速度下可以忽略.

对于线弹性材料杆件(系)，外力P和位移δ为线性关系 → $W=\frac{1}{2}P\delta=U$

例1 能量法求B处的位移




→ $N=\frac{\sqrt{2}}{2}P$

$N_1=-P$

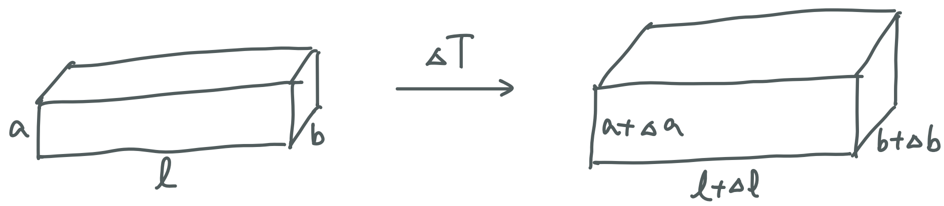
$$U = \underbrace{4\times\frac{1}{2}\frac{N^2 a}{EA} + 1\times\frac{1}{2}\frac{N_1^2\sqrt{2}a}{EA}}_{(1+\frac{\sqrt{2}}{2})\frac{P^2 a}{EA}} = W = \frac{1}{2}P\delta_H \rightarrow \delta_H = (2+\sqrt{2})\frac{Pa}{EA}$$

$\delta_V = 0$ （对称性）

思考：如何处理静不定结构？

## §2.6. 热应变

除应力外，温度变化也会导致材料变形



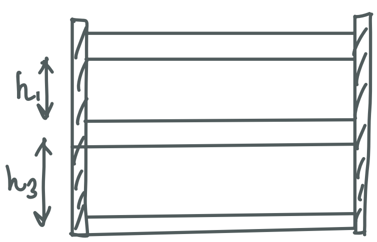

对于各项同性材料 $\varepsilon_a = \frac{\Delta a}{a} = \varepsilon_b = \frac{\Delta b}{b} = \varepsilon_l = \frac{\Delta l}{l} = \alpha\,\Delta T$

材料参数.

在应力和温度的共同作用下，材料的本构关系变为：

$$\boxed{\varepsilon = \frac{\sigma}{E} + \alpha\Delta T}$$

例：




$E_1, A_1, l, \alpha_1$

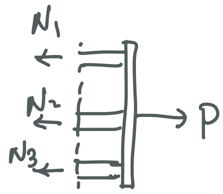
$E_2, A_2, l, \alpha_2$

$E_3, A_3, l, \alpha_3$

在拉力 P，温度变化 $\Delta T$ 下的变形.

What if $l_2 = l + \Delta$ ?

① 平衡方程

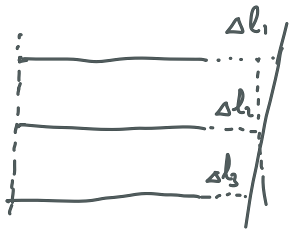



$N_1+N_2+N_3=P$

$N_1h_1+N_3h_3=0$

② 物理 $\Delta l_1=\frac{N_1 l}{E_1A_1}+\alpha_1\Delta T$ , $\Delta l_2=\frac{N_2 l}{E_2A_2}+\alpha_2\Delta T$ , $\Delta l_3=\frac{N_3 l}{E_3A_3}+\alpha_3\Delta T$

③ 几何

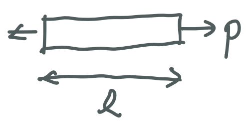



$\frac{\Delta l_1-\Delta l_2}{h_1}=\frac{\Delta l_2-\Delta l_3}{h_3}$ or $\Delta l_2=\frac{h_3}{h_1+h_3}\Delta l_1+\frac{h_1}{h_1+h_3}\Delta l_3$

Check: $h_3=0 \rightarrow \Delta l_2=\Delta l_3$ ✓.

## §2.7 强度与刚度计算

• 刚度：材料/结构抵抗变形的能力.



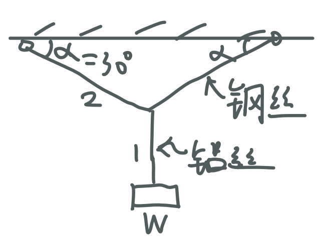

$\Delta l=\frac{Pl}{EA} \rightarrow k=\frac{EA}{l}$. 设计准则为 $u<[u]$

↑ 许用位移

• 强度：材料/结构抵抗破坏的能力.

| | 强度 | 许用应力 | 安全系数 |
|---|---|---|---|
| 脆性材料 | 强度极限 $\sigma_b$ | $[\sigma]=\frac{\sigma_b}{n_b}$ | $n_b=2-5$ |
| 韧性材料 | 屈服极限 $\sigma_s$ | $[\sigma]=\frac{\sigma_s}{n_s}$ | $n_s=1.5-2$ |

例1.



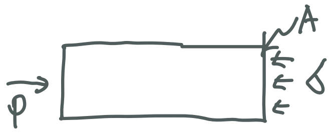

$N_1=W$ , $N_2=\frac{W}{2\sin\alpha}$

$d_1 = 2\ \text{mm}$，$[\sigma_1] = 100\ \text{MPa}$ $\qquad \sigma_1 = \frac{4W}{\pi d_1^2} < [\sigma_1] \to W < 314\ \text{N}$

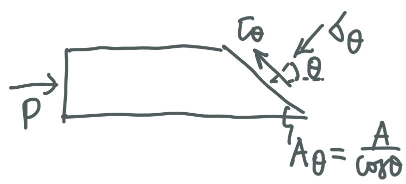
$d_2 = 1\ \text{mm}$，$[\sigma_2] = 240\ \text{MPa}$ $\qquad \sigma_2 = \frac{4W}{2\sin\alpha\, \pi d_2^2} < [\sigma_2] \to W < 188\text{N}$ $\quad \to [W] = 188\text{N}$

可以提高$\alpha$，使得$\sigma_2$减小，$\alpha^* = 56.4^\circ$时，$[W] \to 314\text{N}$.

例2. 现在我们已经初步了解应力. 现考虑一块砖头，其压缩极限强度为$\sigma_c$，剪切极限强度为$\tau_c = \frac{1}{2}\sigma_c$. 如何破坏?




$\to \sigma \equiv \frac{P}{A}$，没有切应力?




$$\Sigma F_x = P - \sigma_\theta \cdot \cos\theta \cdot A_\theta - \tau_\theta \sin\theta\, A_\theta = 0$$

$$\to P - \sigma_\theta A - \tau_\theta \frac{\sin\theta}{\cos\theta} A = 0$$

$$\Sigma F_y = -\sigma_\theta \sin\theta\, A_\theta + \tau_\theta \cos\theta\, A_\theta = 0$$

$$\to \sigma_\theta = \frac{\tau_\theta \cos\theta}{\sin\theta}$$

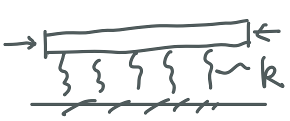
$$\to \boxed{\tau_\theta = \frac{P}{2A}\sin 2\theta \ ,\quad \sigma_\theta = \frac{P}{A}\cos^2\theta}$$

$\sigma_{max} = \sigma_\theta(\theta = 0) = \frac{P}{A}$ ，$\tau_{max} = \tau_\theta(\theta = \frac{\pi}{4}) = \frac{P}{2A}$.

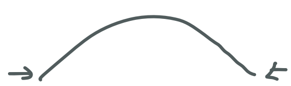
$\hookrightarrow P_{max} = \sigma_c A$ $\qquad \hookrightarrow P_{max} = \tau_c \cdot 2A = \frac{1}{2}\sigma_c A$ ∴剪切破坏

如何表征材料的剪切强度/刚度? —第三章.

<u>稳定性</u>：材料/结构保持原有平衡的能力.




“k很小”：



“k很大”