Thin film fracture

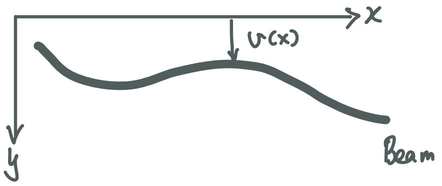
"Thin Film Materials" by L.B. Freund & S. Suresh: Thin films have been inserted into engineering systems in order to accomplish a wide range of practical sevice functions. Among these are micro (nano) electronic devices and packages, MEMS, and surface coating… … To a large extent, the success of this endeavor has been enabled by research leading to reliable means for estimating stress in small material systems and by eartablishing frameworks in which to access the integrity or functionality of the systems. → BVP for thin films!

The PDE

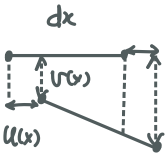
the criteria

Let us first consider a 2D case. We'll show many concepts obtained in 2D systems apply to more general 3D problems.

We consider partially nonlinear kinematics (i.e., moderate rotation) and linear material laws.



Displacement of neutral axis: $u(x)$, $v(x)$



$$u(x+dx) = u(x) + \frac{\partial u}{\partial x} dx + O(dx^2)$$

$$v(x+dx) = v(x) + \frac{\partial v}{\partial x} dx + O(dx^2)$$

$$(dx')^2 = [dx + u(x+dx) - u(x)]^2 + [v(x+dx) - v(x)]^2$$

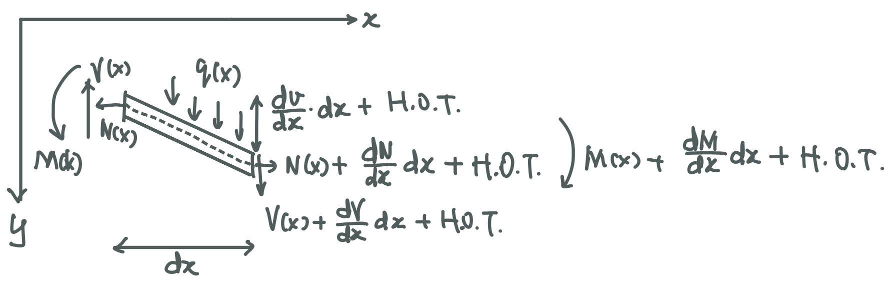
$$= \left(dx + \frac{\partial u}{\partial x} dx\right)^2 + \left(\frac{\partial v}{\partial x} dx\right)^2$$

$$\rightarrow dx' = dx\sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

2D-finite strain: $\varepsilon_{xx} = \dfrac{dx' - dx}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left[\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2\right]$

Small strain: $\dfrac{\partial u}{\partial x} \ll 1 \rightarrow \left(\dfrac{\partial u}{\partial x}\right)^2 \ll \dfrac{\partial u}{\partial x}$

Moderate rotation: $\left(\dfrac{\partial v}{\partial x}\right)^2 \sim \dfrac{\partial u}{\partial x} \ll 1 \rightarrow \boxed{\varepsilon_{xx} = \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x}\right)^2}$



$$\sum F_x = 0 \rightarrow \frac{dN}{dx} = 0 \rightarrow N = \text{constant}$$

$$\sum F_y = 0 \rightarrow -V(x) + q\,dx + V(x) + \frac{dV}{dx}dx = 0 \rightarrow \frac{dV}{dx} = -q$$

$$\sum M_z^{x+dx} = 0 \rightarrow -\cancel{M(x)} + V\cdot dx - N\cdot\left(\frac{dv}{dx}dx + \cancelto{0}{O(d^2x)}\right) - \cancelto{0}{q\cdot O(d^2x)} + \cancel{M(x)} + \frac{dM}{dx}dx + \cancelto{0}{HOT} = 0$$

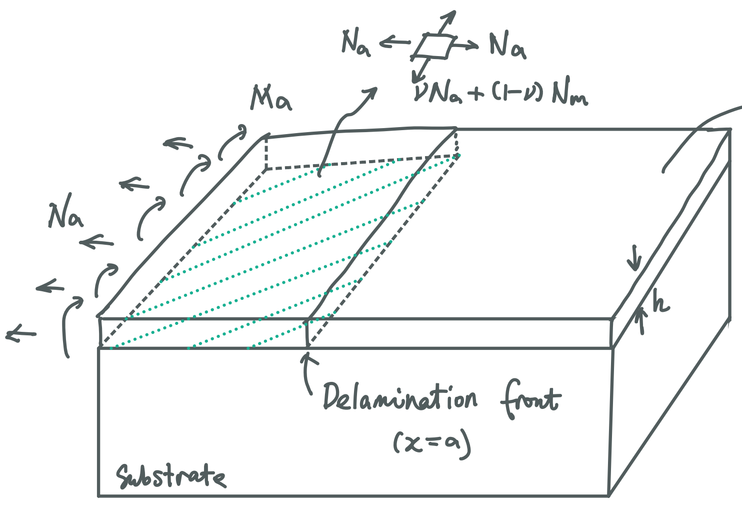
$$\rightarrow \frac{dM}{dx} - N\frac{dv}{dx} + V = 0 \rightarrow \frac{d^2M}{dx^2} - N\frac{d^2v}{dx^2} - q = 0$$

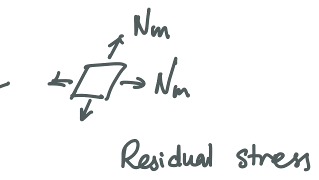
Finally, linear material law gives $M = BK$, $K = \frac{v''}{(1+v'^2)^{3/2}} \simeq v''$ for moderate rotations.

$$\therefore \boxed{\frac{d^4v}{dx^4} - \frac{N}{B}\frac{d^2v}{dx^2} = \frac{q}{B}}$$

, where $N = E'h\varepsilon_{xx}$, $B = \frac{1}{12}E'h^3$, $E' = \frac{E}{1-\nu^2}$ in general.

We are interested in the energy release rate in this system. Consider a region that is close to the edge of the delamination zone. At this level of observation, the edge is essentially straight and the state of deformation is "generalized" plane strain.

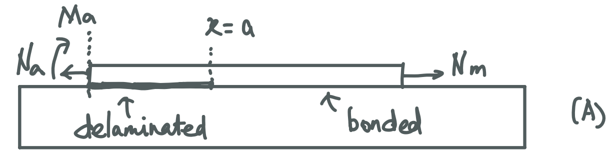


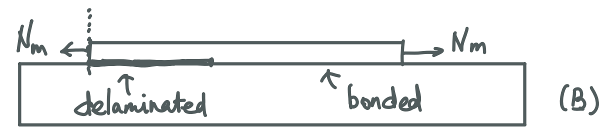


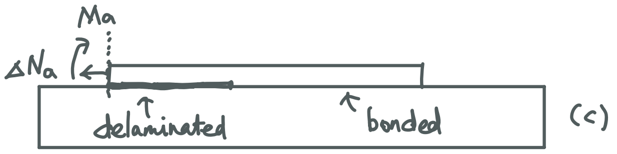
At $x=a$,

$$\varepsilon_{yy}^{+} = \frac{1-\nu}{Eh}N_m$$

$$\varepsilon_{yy}^{-} = \frac{\cancel{\nu N_a} + (1-\nu)N_m - \cancel{\nu N_a}}{Eh} = \varepsilon_{yy}^{+}$$







$\Delta N_a = N_a - N_m$

The state (A) is the superposition of states (B) and (c).

- The driving force for delamination in (B) is zero.
- $\Pi = U_{SE} - W_\sigma$.

$$a \to a+\delta a, \begin{cases} \delta U_{SE} = \frac{1}{2}\frac{\Delta N_a^2}{E'h}\delta a + \frac{1}{2}\frac{M_a^2}{B}\delta a \\ \delta W_\sigma = 2\,\delta U_{SE} \end{cases}$$

$$\rightarrow \mathcal{G} = -\frac{\delta\Pi}{\delta a} = \delta U_{SE}/\delta a$$

$$\therefore \boxed{\mathcal{G}(a) = \frac{1-\nu^2}{2Eh}\Delta N_a^2 + \frac{1}{2B}M_a^2}$$ "Local condition"

The energy release rate for advance of the delamination front is determined by the edge loads, ΔN_a and M_a, which are <u>not known</u> a priori in general (Need to solve the BVP).

According to Hutchison & Suo (1991), the stress intensity factors are

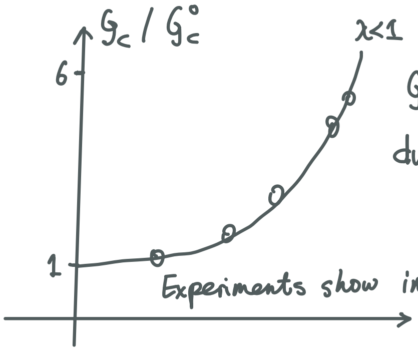
$$K_I = \frac{1}{\sqrt{2}}\left[\Delta N_a h^{-1/2}\cos\omega + 2\sqrt{3}\,M_a h^{-\frac{3}{2}}\sin\omega\right]$$

$$K_{II} = \frac{1}{\sqrt{2}}\left[\Delta N_a h^{-1/2}\sin\omega - 2\sqrt{3}\,M_a h^{-\frac{3}{2}}\cos\omega\right]$$

$$\rightarrow \underset{\text{Mode-Mixity}}{\tan\psi} = \frac{\mathrm{Im}(Kh^{i\varepsilon})}{\mathrm{Re}(Kh^{i\varepsilon})} = \frac{\sqrt{12}\,M_a + \Delta N_a h\tan\omega}{-\sqrt{12}\,M_a\tan\omega + \Delta N_a h}$$

$$\omega = \omega(h/\underset{\text{Substrate thickness}}{H}, \varepsilon) \to 45^\circ - 65^\circ \quad \text{as } h/H \to 0$$

where $K = K_I + iK_{II}$, $\varepsilon = \varepsilon(E, \nu, \underbrace{E_s, \nu_s}_{\text{Substrate properties}})$

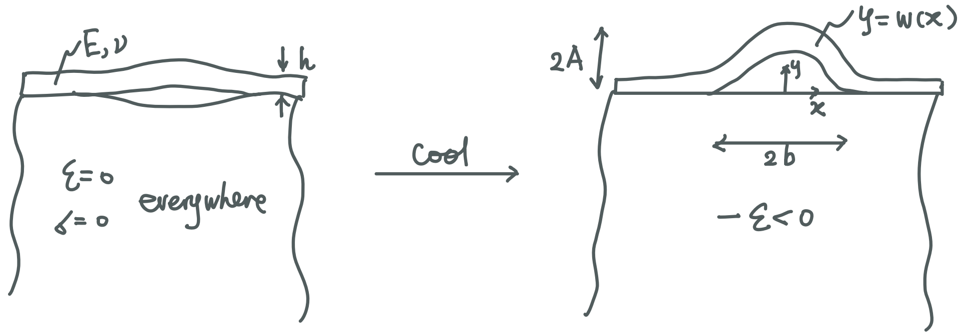


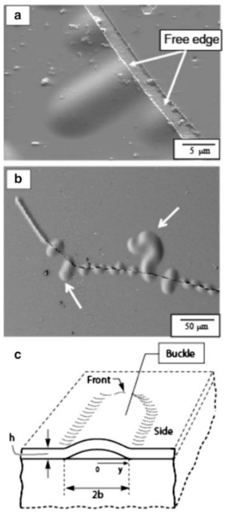
$$\mathcal{G}_c(\psi) = \mathcal{G}_c^1\left[1 + (\lambda-1)\sin^2\psi\right]^{-1}$$

due to asperity contact and plasticity.

Experiments show increased $\mathcal{G}_c$ as K_{II} increases.

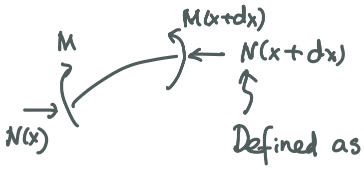
Buckle delamination





When to occur and what determines A & b?

Let us solve for this boundary value problem :



$$B\frac{d^4w}{dx^4} + N\frac{d^2w}{dx^2} = 0 \quad \& \quad N = \text{constant}$$

$$\rightarrow w = A + \underbrace{\cancel{C}}_{0}x + \underbrace{\cancel{D}}_{0} \sin\sqrt{\tfrac{N}{D}}\,x + F\cos\sqrt{\tfrac{N}{B}}\,x.$$

symmetry

Boundary conditions :

$$w(\pm b) = 0 \rightarrow A + F\cos\sqrt{\tfrac{N}{B}}\,b = 0 \longrightarrow F = A$$

$$w'(\pm b) = 0 \rightarrow \sin\sqrt{\tfrac{N}{B}}\,b = 0 \rightarrow N = \frac{\pi^2 B}{b^2} \quad \left(\text{Recall the Euler instabity } P_{cr} = \frac{\pi^2 EI}{(\mu l)^2}\right)$$

To determine A, we need to describle the axial strain of the centerline ε_{xx}

$$\varepsilon_{xx} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 = \frac{N}{E'h}$$

$$\underbrace{u(x=b) - u(x=-b)}_{\substack{-2b\cdot\varepsilon \\ \text{due to cooling}}} = \int_{-b}^{b}\frac{du}{dx}dx = \underbrace{\int_{-b}^{b}\left[\frac{N}{E'h} - \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]dx}$$

$$= \underbrace{\frac{N}{E'h}}_{\varepsilon_c = \frac{\pi^2}{12}\left(\frac{h}{b}\right)^2}\cdot 2b - \underbrace{\int_{-b}^{b}\frac{N}{B}A^2\sin\sqrt{\tfrac{N}{B}}\,x\,dx}_{\frac{\pi^2}{2b^2}A^2\int_{-b}^{b}\sin^2\left(\frac{\pi}{b}x\right)dx \;=\; \frac{\pi^2}{2b}A^2}$$

$$\rightarrow \quad -2b\varepsilon = 2b\varepsilon_c - \frac{\pi^2}{2b}A^2 \rightarrow A^2 = \frac{4b^2}{\pi^2}(\varepsilon - \varepsilon_c)$$

↖ critical strain for buckling.

$$\therefore \boxed{w(x) = A\left(1 + \cos\frac{\pi}{b}x\right), \quad A = \frac{2b}{\pi}(\varepsilon - \varepsilon_c)^{1/2}, \quad \varepsilon_c = \frac{\pi^2}{12}\left(\frac{h}{b}\right)^2}$$

- Now, we know the solution for buckled film. Let's compute the energy release rate.

$$U_{flat} = \frac{1}{2} E' \varepsilon^2 h (l-2b) \quad \text{, where } l \text{ is the total length of the film}$$

$$U_{buckle} = \int_{-b}^{b} \frac{1}{2} B (w'')^2 + \frac{1}{2} E'h (\varepsilon_{xx})^2 \, dx$$

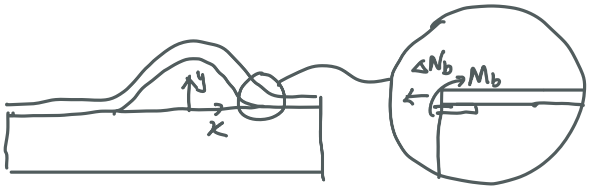
$$= \frac{1}{2} \cdot \frac{1}{12} E' h^3 \cdot \frac{\pi^4}{b^4} \cdot A^2 \underbrace{\int_{-b}^{b} \cos^2 \frac{\pi}{b} x \, dx}_{b} + \frac{1}{2} E'h \frac{\overset{\frac{\pi^4}{b^4} \cdot \frac{1}{144} E'^2 h^6}{N^2}}{(E'h)^2} \cdot 2b$$

$$= \underbrace{\frac{1}{24} E' h^3 \frac{\pi^4}{b^3} \cdot \frac{4b^2}{\pi^2}}_{\frac{\pi^2}{6} E'h \cdot \frac{h^2}{b}} (\varepsilon - \varepsilon_c) + \frac{\pi^4}{144} \frac{E'h}{b^3} \cdot h^4$$

$$\rightarrow \quad U_{SE} = \frac{E'h}{2} \left[(l-2b) \varepsilon^2 + \frac{\pi^2}{3} \frac{h^2}{b} \varepsilon \underbrace{- \frac{\pi^2}{3} \frac{h^2}{b} \cdot \frac{\pi^2}{12} \left(\frac{h}{b}\right)^2 + \frac{\pi^4}{72} \frac{h^4}{b^3}}_{-\frac{\pi^4}{72} \frac{h^4}{b^3}} \right]$$

$$\mathcal{G} = - \frac{\partial U_{SE}}{\partial (2b)} = \frac{E'h}{2} \left(\varepsilon^2 + \underbrace{\frac{\pi^2}{6} \frac{h^2}{b^2} \varepsilon}_{2 \varepsilon \varepsilon_c} - \underbrace{\frac{\pi^4}{48} \frac{h^4}{b^4}}_{3 \varepsilon_c^2} \right) = \frac{1}{2} E'h \left(\varepsilon + 3 \varepsilon_c \right) \left(\varepsilon - \varepsilon_c \right)$$

- We can also obtain this according to the local observation

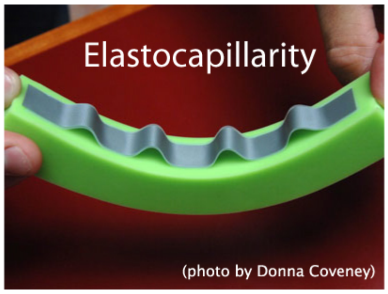


$$\Delta N_b = (-N) - (-E'h\varepsilon) = -\frac{\pi^2 B}{b^2} + E'h\varepsilon \; ; \qquad M_b = B w''(x=b) = +B A \frac{\pi^2}{b^2}$$

$$\text{(P90)} \rightarrow \mathcal{G} = \underbrace{\frac{1}{2E'h} \Delta N_b^2}_{\mathcal{G}_s} + \underbrace{\frac{1}{2B} M_b^2}_{\mathcal{G}_b} = \frac{E'h}{2} \left(\varepsilon^2 + \frac{\pi^2}{6} \frac{h^2}{b^2} \varepsilon - \frac{\pi^4}{48} \frac{h^4}{b^4} \right) \checkmark = \frac{\pi^4}{96} \frac{E'h (3A^4 + 4A^2 h^2)}{b^4} \, ?$$

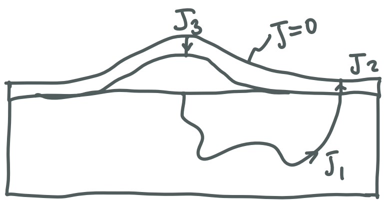
As the substrate is very soft (Pan et al IJSS, 2014), or the interface is slippery (Dai et al. JMPS, 2020) so that $\Delta N_b \rightarrow 0$, $\mathcal{G}_s$ is not important:

$$\Gamma = \frac{1}{2B} M_b^2 = \frac{\pi^4}{2} \frac{BA^2}{b^4} \xrightarrow[2b=\lambda]{2A=\delta} 2\pi^4 \frac{B\delta^2}{\lambda^4} \quad \text{(D. Vella et al. PNAS 2009)}$$



(photo by Donna Coveney)

- Lastly, let's try J integral.



$$J = \int_\Gamma W n_1 - \sigma_{ij} n_j u_{i,1}\, d\Gamma$$

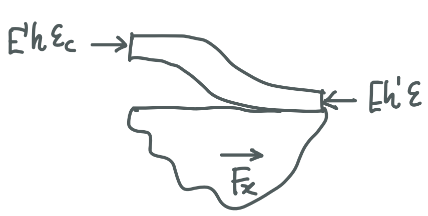
$$J_2 = \int_0^h W - \sigma_{xx}\varepsilon_{xx}\, dy = -\frac{1}{2}E'h\varepsilon^2$$

$$J_3 = \int_{-h}^{0} \overset{n_1=-1}{-W + \sigma_{xx}\varepsilon_{xx}}\, d(-y)$$

$$= \int_0^h \frac{1}{2}\sigma_{xx}\varepsilon_{xx}\, dy = \int_0^h \frac{1}{2}\left(\overset{\text{stretching}}{\sigma_{xx}^{s}} + \overset{\text{bending}}{\sigma_{xx}^{b}}\right)\left(\varepsilon_{xx}^{s} + \varepsilon_{xx}^{b}\right) dy$$

$$= \frac{1}{2}B(w'')^2 + \frac{1}{2}E'h\varepsilon_{xx}^2 \leftarrow = \frac{N}{E'h} = \varepsilon_c$$

$$= \frac{1}{24}E'h^3 \underbrace{\frac{4b^2}{\pi^2}(\varepsilon-\varepsilon_c)}_{A^2} \cdot \frac{\pi^4}{b^4} + \frac{1}{2}E'h\varepsilon_c^2 = \frac{1}{2}E'h\left(4\varepsilon\varepsilon_c - 3\varepsilon_c^2\right)$$



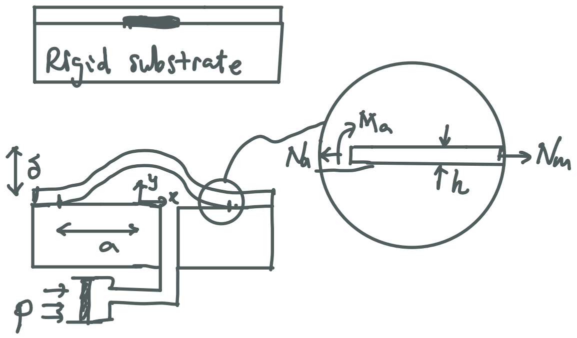
$$J_1 = \int_{\Gamma_1} \cancelto{0,\ \text{rigid}}{W n_1} - t_x \cancelto{-\varepsilon}{\varepsilon_{xx}} - t_y \cancelto{0\ \text{rigid}}{\varepsilon_{xy}}\, d\Gamma$$

$$= \varepsilon\int_\Gamma t_x\, d\Gamma = \varepsilon F_x = Eh'(\varepsilon^2 - \varepsilon\varepsilon_c)$$

$$\rightarrow G = J = J_1 + J_2 + J_3 = \frac{Eh'}{2}\left(-\varepsilon^2 + 4\varepsilon\varepsilon_c - 3\varepsilon_c^2 + 2\varepsilon^2 - 2\varepsilon\varepsilon_c\right) = \frac{E'h}{2}\left(\varepsilon^2 + 2\varepsilon\varepsilon_c - 3\varepsilon_c^2\right) \checkmark$$

Pressurized bulge of uniform width

The straight-sided bulge configuration is perhaps of less practical significance than the circular case. But the mechanical response of the film can be described in a fairly "transparent" way at various levels of approximations — useful for introducing ideas.



Now the deflection results from external loading p (positively defined upward)

$$B\frac{d^4w}{dx^4} - N\frac{d^2w}{dx^2} = p$$

There are three sources of elastic energy: bending, stretching & residual stress. Let us consider a scaling argument.

Geometry: $K \sim \delta/a^2$, $\varepsilon \sim \delta^2/a^2$

Bending energy: $U_b \sim BK^2 \sim B\delta^2/a^4$ (per area)

Stretching energy: $U_s \sim N\varepsilon \sim (E'h\varepsilon + N_m)\varepsilon \sim \begin{cases} E'h\delta^4/a^4 & \text{as } \frac{N_m}{E'h} \ll \frac{\delta^2}{a^2} \quad \text{(Membrane)} \\ N_m\delta^2/a^2 & \text{as } \frac{N_m}{E'h} \gg \frac{\delta^2}{a^2} \quad \text{(Pretension)} \end{cases}$

- ### The bending response ($U_b \gg U_s$)

This case leads to the simplest level of approximation — linear plate theory

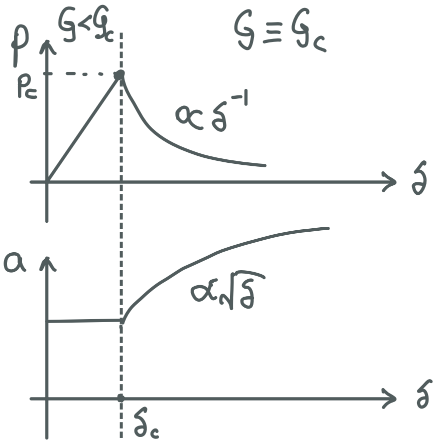
$$B\frac{d^4w}{dx^4} = p$$

$$w(x=\pm a) = 0 \quad \Rightarrow \quad w(x) = \frac{pa^4}{24B}\left(1-\frac{x^2}{a^2}\right)^2 \quad \& \quad p = \frac{24B\delta}{a^4} \quad (\text{linear } p\text{–}\delta \text{ relation})$$

$$w'(x=\pm a) = 0$$

Accurate when $B\delta^2/a^4 \gg \{E'h\delta^4/a^4,\ N_m\delta^2/a^2\}$, i.e., $\delta \ll \left(\frac{B}{E'h}\right)^{1/2} \sim h$ and $\frac{N_m}{E'h} \ll \frac{B}{E'ha^2} \sim \frac{h^2}{a^2}$

This "configurational" driving force for delamination at the edge of the pressurized zone can be calculated by Eq on P90 with $\Delta N_a = 0$



$$\mathcal{G} = \underbrace{\frac{1}{2E'h}\Delta N_a^2}_{\sim E'h\frac{\delta^4}{a^4}} + \underbrace{\frac{1}{2}B(w'')_a^2}_{\sim E'h^3\frac{\delta^2}{a^4}} = \frac{1}{18}\frac{p^2a^4}{B} = \frac{32B\delta^2}{a^4}$$

$$\sim E'h\frac{\delta^4}{a^4} \ll \sim E'h^3\frac{\delta^2}{a^4}$$

- "Large deflection" response ($U_b \sim U_s$)

If the center point deflection δ increases to values on the order of h, we need to consider the generated membrane stress in the film due to transverse deflection (in addition to residual membrane stress). Here we consider a simplified case in which $N_m = 0$.

$$B\frac{d^4w}{dx^4} - N\frac{d^2w}{dx^2} = p \longrightarrow w(x) = -\frac{p}{2N}x^2 + A + \overset{0}{\cancel{B}}x + \overset{0}{\cancel{C}}\sinh\sqrt{\frac{N}{B}}x + D\cosh\sqrt{\frac{N}{B}}x$$

due to symmetry

Boundary conditions : $w(x=\pm a) = w'(x=\pm a) = 0$

$$\left.\begin{aligned} -\frac{p}{2N}a^2 + A + D\cosh\sqrt{\frac{N}{B}}a = 0 \\ -\frac{p}{N}a + D\sqrt{\frac{N}{B}}\sinh\sqrt{\frac{N}{B}}a = 0 \end{aligned}\right\} \xrightarrow{\xi = \left(\frac{Na^2}{B}\right)^{1/2}} \begin{aligned} A &= \frac{pa^4}{2B}\frac{\xi - 2\coth\xi}{\xi^3} \\ D &= \frac{pa^4}{B}\frac{1}{\xi\sinh\xi} \end{aligned}$$

We obtain $w(x) = \frac{pa^4}{B}\left[\frac{\xi - 2\coth\xi}{2\xi^3} - \frac{1}{\xi^2}\left(\frac{x}{a}\right)^2 + \frac{\cosh\left(\frac{\xi x}{a}\right)}{\xi^3\sinh\xi}\right]$

$$\rightarrow \delta = \frac{pa^4}{B}\left(\frac{\xi - 2\coth\xi}{2\xi^3} + \frac{1}{\xi^3\sinh\xi}\right) = \begin{cases} \frac{pa^4}{24B}\left[1 - \frac{1}{10}\xi^2 + O(\xi^4)\right], & \text{for } \xi \ll 1 \text{ (Plate response)} \\ \frac{pa^2}{2N}\left[1 - \frac{2\coth\xi}{\xi} + O(e^{-\xi}\xi^{-3})\right], & \text{for } \xi \gg 1 \text{ (Membrane response)} \end{cases}$$

What is membrane response as $\tau \gg 1$? Imagine zero-bending modulus plate: $-N\frac{d^2w}{dx^2}=P$, its solution is simply $w=\frac{Pa^2}{2N}\left(1-\frac{x^2}{a^2}\right)$. It satisfies $w(\pm a)=0$ but not $w'(\pm a)=0$!

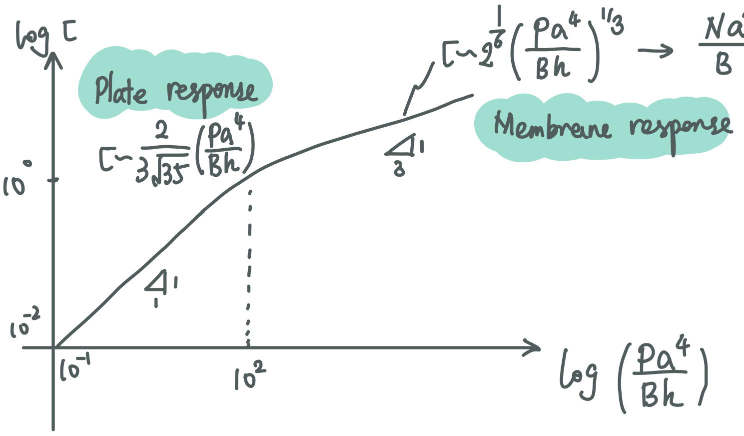
Note that we still don't know what N or τ is! Need to use BCs about in-plane displacement.

$$\frac{N}{E'h}=\varepsilon_{xx}=\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2 \rightarrow \frac{du}{dx}=\frac{N}{E'h}-\frac{1}{2}\left(\frac{dw}{dx}\right)^2$$

$$\rightarrow u(a)-u(-a)=\overset{\text{since } N_m=0}{\overset{0}{\cancel{\varepsilon_m}}}\cdot 2a=\int_{-a}^{a}\left[\frac{B\tau^2}{E'ha^2}-\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]dx$$

$$=\frac{h^2}{6a}\tau^2-\frac{p^2a^7}{6B^2}\,\frac{12+2\tau^2-9\tau\coth\tau-3\tau^2/\sinh^2\tau}{\tau^6}$$

$$\rightarrow \tau^8=\left(\frac{pa^4}{Bh}\right)^2\left(12+2\tau^2-9\tau\coth\tau-3\tau^2/\sinh^2\tau\right) \quad \text{or} \quad \frac{pa^4}{Bh}=f(\tau)$$



$$\tau\sim 2^{\frac{1}{6}}\left(\frac{Pa^4}{Bh}\right)^{1/3}\rightarrow \frac{Na^2}{B}\sim 2^{1/3}\left(\frac{Pa^4}{Bh}\right)^{2/3}\rightarrow N\sim\frac{p^{2/3}a^{2/3}(E'h)^{1/3}\cancel{h^{2/3}}}{\cancel{h^{2/3}}\,6^{1/3}}$$

- When $\frac{Pa^4}{Bh}\sim\frac{\delta}{h}\ll 1$, $\tau\to 0$, $\delta=\frac{Pa^4}{24B}$ or $P=\frac{24B}{a^4}\delta$ (Plate)
- When $\frac{Pa^4}{Bh}\sim\frac{\delta}{h}\gg 1$, $\delta=\frac{Pa^2}{2N}=\frac{6^{1/3}}{2}\frac{P^{1/3}a^{\frac{4}{3}}}{(E'h)^{1/3}}=\left(\frac{3Pa^4}{4E'h}\right)^{1/3}$ or $P=\frac{4}{3}\frac{E'h}{a^4}\delta^3$ (Membrane)

Now we are able to determine ΔN_a and M_a in terms of τ, specifically

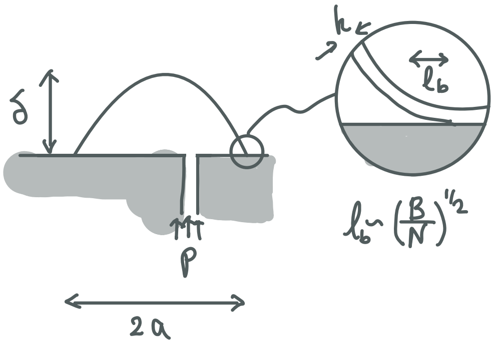
$$\Delta N_a=N=\frac{B\tau^2}{a^2},\quad M_a=Bw''(x=a)=\frac{Pa^2}{\tau^2}(1-\tau\coth\tau)=\frac{Bh}{a^2}\underbrace{f(\tau)\,\frac{1-\tau\coth\tau}{\tau^2}}_{=g(\tau)}$$

$$G=\frac{1}{2}\frac{\Delta N_a^2}{Eh}+\frac{1}{2}\frac{M_a^2}{B}=\frac{B^2}{2E'ha^4}\tau^4+\frac{Bh^2}{2a^4}g^2(\tau)=\frac{E'h^5}{a^4}\frac{\tau^4}{288}\left[1+\frac{12(1-\tau\coth\tau)^2}{2(6+\tau^2)-9\tau\coth\tau-3\tau^2\operatorname{csch}^2\tau}\right]$$

$$\sim \begin{cases} \frac{E'h^5}{a^4} \cdot \frac{35}{96} \tau^2 = \frac{Bh^2}{a^4} \frac{35}{8} \cdot \frac{4}{9 \cdot 35} \frac{p^2 a^8}{B^2 h^2} = \frac{p^2 a^4}{18B} \text{, as } \tau \ll 1 & \text{(Plate limit)} \checkmark \\ \frac{E'h^5}{a^4} \cdot \frac{7}{288} \tau^4 = \frac{7}{2 \times 6^{2/3}} \left(\frac{p^4 \cdot a^4}{E'h}\right)^{1/3} \text{, as } \tau \gg 1 & \text{(Membrane limit? Need to check)} \end{cases}$$

• Membrane response ($U_s \gg U_b$)

Still consider $N_m = 0$ so that $U_s \gg U_b$ means $E'h \frac{\delta^4}{a^4} \gg B \frac{\delta^2}{a^4}$, i.e., $\delta \gg \sqrt{\frac{B}{E'h}} \sim h$



Need to be careful about l_b – two ways to analyze:

Prescribed p or prescribed δ. Let's do the latter.

$$\varepsilon_{xx} \sim \frac{\delta^2}{a^2} \rightarrow N \sim E'h \frac{\delta^2}{a^2} \rightarrow l_b \sim \left(\frac{B}{E'h\delta^2/a^2}\right)^{1/2} \sim \frac{h}{\delta} a \ll a$$

Non-dimensionalization

$$X = \frac{x}{a}, \quad W = \frac{w}{\delta}, \quad \breve{N} = \frac{N}{E'h\delta^2/a^2}, \quad P = \frac{p}{E'h\delta^2/a^2 \times \delta/a^2} = \frac{pa^4}{E'h\delta^3}$$

$$B \frac{d^4 w}{dx^4} - N \frac{d^2 w}{dx^2} = p \longrightarrow \frac{B \cdot \delta}{a^4} W_{XXXX} - \breve{N} \frac{E'h\delta^2}{a^2} \cdot \frac{\delta}{a^2} W_{XX} = P \cdot \frac{E'h\delta^3}{a^4}$$

$$\therefore \epsilon^2 W_{XXXX} - \breve{N} W_{XX} = P \text{, where } \epsilon = \left(\frac{B}{E'h\delta^2}\right)^{1/2} \sim \frac{h}{\delta} \ll 1$$

This definition gives $l_b \sim \epsilon a$

Since $\epsilon^2 \lll 1$, we neglect the high order term and immediately have the solution:

$$W = \frac{P}{2\breve{N}}(1 - X^2) \quad \text{or} \quad w(x) = \underbrace{\frac{pa^2}{2N}}_{= \delta \text{ since } w(0) = \delta} \left(1 - \frac{x^2}{a^2}\right) \text{ in dimensional form}$$

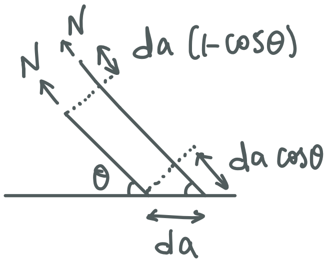
where we have used $W(\pm 1) = 0$. To calulate $\breve{N}$, recall that

$$\frac{N}{E'h} = \varepsilon_{xx} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \rightarrow u(a) - u(-a) = 0 = \int_{-a}^{a} \left[\frac{N}{E'h} - \frac{1}{2}\left(2\frac{\delta x}{a^2}\right)^2\right] dx$$

$\rightarrow N = \frac{2}{3} E'h \frac{\delta^2}{a^2}$ and $p = \frac{2N\delta}{a^2} = \frac{4}{3} E'h \frac{\delta^3}{a^4}$ (Agree with results on Page 96)

What is the energy release rate? Two perspectives!

① From the point of view of $-\frac{\partial \Pi}{\partial a}$. (Kendall's peeling angle)



(Local view of membrane)

$$d\Pi = dU_{SE} - dW_s$$

$$= da \times \frac{1}{2}\frac{N^2}{E'h} - N \cdot da(1-\cos\theta) - N \cdot \overset{\text{Stretching}}{\overbrace{\frac{N}{E'h}}} da$$

$$\rightarrow \boxed{\mathcal{G} = -\frac{d\Pi}{da} = N(1-\cos\theta) + \frac{1}{2}\frac{N^2}{E'h}} \quad \circledast$$

- θ is constant if delamination occurs
- Contribution by p is neglected since the level of observation is small

$$= \frac{1}{2} w'^2 \Big|_a N + \frac{1}{2}\frac{N'}{E'h}$$

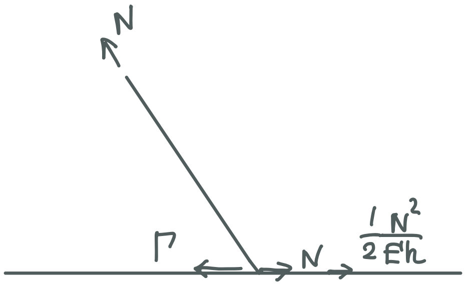
$$= \frac{1}{2} \cdot \left(\frac{2\delta}{a}\right)^2 \cdot \frac{2}{3} E'h \frac{\delta^2}{a^2} + \frac{1}{2} \cdot E'h \cdot \left(\frac{2}{3}\frac{\delta^2}{a^2}\right)^2$$

$$= \frac{14}{9} E'h \frac{\delta^4}{a^4}$$

$$= \frac{14}{9} E'h \frac{1}{a^4} \left(\frac{3pa^4}{4E'h}\right)^{\frac{4}{3}}$$

$$= \frac{7}{2 \times 6^{2/3}} \left(\frac{p^4 \cdot a^4}{E'h}\right)^{1/3}$$

(Agree with results on Page 97)



Geometric interpretation of $\circledast$

It should be noted here that $\circledast$ applies to the peeling problem with ang θ.

In particular, when θ is not too close to 0°, $N >> \frac{N^2}{E'h}$, we have $\mathcal{G} = N(1-\cos\theta)$, i.e., peeling at $\theta = \frac{\pi}{2}$, $\mathcal{G}_c = P_c$.

This result is neat and nice, but it does not give anything at K_I, K_{II} or ψ, which needs information at ΔN_a and M_a.

② From the point of view of boundary layer analysis

Want to understand what is going on near $x=\pm a$. Return the unsimplified equation at the level of observation $\sim l_b$, i.e., ϵ in dimensionless form.

$$\xi = \frac{x-1}{\epsilon}, \quad \frac{d}{dx} = \frac{d}{d\xi}\cdot\frac{1}{\epsilon} \rightarrow \frac{\epsilon^2}{\epsilon^4} W_{\xi\xi\xi\xi} - \frac{\tilde{N}}{\epsilon^2} W_{\xi\xi} = P \rightarrow W_{\xi\xi\xi\xi} - \tilde{N} W_{\xi\xi} = P\cancelto{0}{\epsilon^2}$$

(pressure not important here)

The solution is

$$W(\xi) = C_1 + C_2\xi + C_3 e^{+\tilde{N}^{1/2}\xi} + \cancelto{0}{C_4} e^{-\tilde{N}^{1/2}\xi}$$

Boundary and matching conditions:

$$W_x(x\rightarrow 1) = \frac{-P}{\tilde{N}} = W_\xi \cdot \frac{1}{\epsilon}$$

- At $x \rightarrow a$, $\xi \rightarrow 0$, $W = W' = 0$
- At $x \rightarrow a^-$, $\xi \rightarrow -\infty$, W' finite & $W' \rightarrow -\frac{\epsilon P}{\tilde{N}} = C_2$

$$\rightarrow W(\xi) = -\frac{P\epsilon}{\tilde{N}}\xi + \frac{P\epsilon}{\tilde{N}^{3/2}}\left(e^{\tilde{N}^{1/2}\xi} - 1\right)$$

$$W''(\xi) = \frac{P\epsilon}{\tilde{N}^{1/2}} e^{\tilde{N}^{1/2}\xi} = \frac{4/3}{\sqrt{2/3}}\epsilon\, \mathrm{Exp}\left[\sqrt{\frac{2}{3}}\frac{(x-1)}{\epsilon}\right]$$

$w'' \times \frac{a^2}{\delta} \times \epsilon^2$

from leading order solution

(By a few $\epsilon = \frac{h}{\delta}$ away from the edge, the curvature decays to zero)

Therefore, $\Delta N_a = \frac{2}{3} E'h \frac{\delta^2}{a^2}$,

$$M_a = B w''(x=a) = B\cdot\frac{\delta}{a^2}\frac{1}{\epsilon^2}\cdot W''(\xi=0) = B\frac{\delta}{a^2}\left(\frac{E'h\delta^2}{B}\right)^{1/2}\frac{4}{6^{1/2}} = \left(\frac{8}{3}\frac{BE'h\delta^4}{a^4}\right)^{1/2}$$

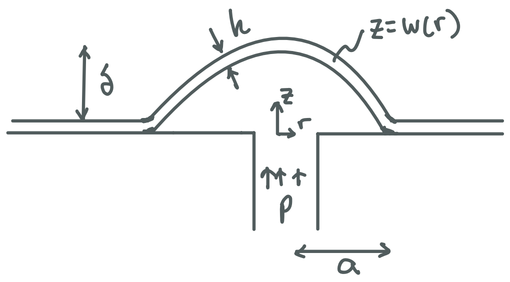
$$\rightarrow \mathcal{G} = \frac{1}{2}\frac{\Delta N_a^2}{E'h} + \frac{1}{2}\frac{M_a^2}{B} = \frac{2}{9}E'h\frac{\delta^4}{a^4} + \frac{8}{2\times 3}E'h\frac{\delta^4}{a^2} = \frac{14}{9}E'h\frac{\delta^4}{a^2} \quad ✓$$

$$\tan\psi = \frac{\sqrt{12}M_a + \Delta N_a h \tan\omega}{-\sqrt{12}M_a\tan\omega + \Delta N_a h} = \frac{\sqrt{6} + \tan\omega}{1 - \sqrt{6}\tan\omega} \sim -1.2 - -0.8 \text{ for } 45° < \omega < 65°$$

Jensen (1993)

(100)
Finally

Circular pressurized bulge



Let's still focus on linear material law and moderate rotation.

In the axisymmetric configuaration, there's a pair of equilibrium equations (see Mansfield, 2005)

Out of plane equilibrium equation:

$$\underbrace{\nabla^2(B\nabla^2 w)}_{\text{Bending}} - \underbrace{(N_{rr}K_{rr} + N_{\theta\theta}K_{\theta\theta})}_{\text{Stretching}} = p \quad , \quad \underbrace{K_{rr} = \frac{d^2w}{dr^2} \quad , \quad K_{\theta\theta} = \frac{1}{r}\frac{dw}{dr}}_{\text{Curvatures}}$$

In plane equilibrium equation:

$$\frac{dN_{rr}}{dr} + \frac{N_{rr} - N_{\theta\theta}}{r} = 0$$

Material law: $\varepsilon_{rr} = \frac{1}{Eh}(N_{rr} - \nu N_{\theta\theta})$, $\varepsilon_{\theta\theta} = \frac{1}{Eh}(N_{\theta\theta} - \nu N_{rr})$

or

$$N_{rr} = \frac{Eh}{1-\nu^2}(\varepsilon_{rr} + \nu\,\varepsilon_{\theta\theta}) \, , \quad N_{\theta\theta} = \frac{Eh}{1-\nu^2}(\varepsilon_{\theta\theta} + \nu\,\varepsilon_{rr})$$

Kinematic relations: $\varepsilon_{rr} = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2$, $\varepsilon_{\theta\theta} = \frac{u}{r}$

Boundary conditions: $\frac{dw}{dr} = \frac{d^3w}{dr^3} = 0$, $u = 0$ at $r = 0$ (symmetry)

$w = \frac{dw}{dr} = 0$, $u = \frac{N_m(1-\nu)}{Eh}$ at $r = a$

↖ Residual stress.

Energy release rate: $\mathcal{G} = -\frac{\partial \Pi}{\partial A}$ ("global")

$= \frac{1-\nu^2}{2Eh}\Delta N_a^2 + \frac{1}{2B}M_a^2$ ("local")

Let's examine the elastic energy (density) associated with bending, induced tension and residual tension.

- $U_b \sim B\kappa^2 \sim B\left(\frac{\delta}{a^2}\right)^2 \sim B\frac{\delta^2}{a^4}$ (Bending)
- $U_s \sim Eh\varepsilon_s^2 \sim Eh\left(\frac{\delta}{a}\right)^4 \sim Eh\frac{\delta^4}{a^2}$ (Induced tension)
- $U_p \sim N_m\varepsilon_s \sim N_m\left(\frac{\delta}{a}\right)^2 \sim N_m\frac{\delta^2}{a^2}$ (Residual tension)

The system can be linearized as long as U_s is not important since only this term involves nonlinear kinematics. For example, when $U_p \gg U_s$, i.e., $Eh\frac{\delta^2}{a^2} \ll N_m$, both N_{rr} and $N_{\theta\theta}$ approach N_m. The in-plane equation is satisfied automatically, and the out of plane equation becomes

$$B\nabla^4 w - N_m\nabla^2 w = p. \qquad (*)$$

The solution can be readily obtained: $w(r) = \underbrace{\frac{pa^2}{4N_m}\left(1-\frac{r^2}{a^2}\right)}_{\text{Particular sol.}} + w_h(r)$

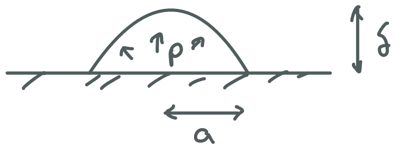
To seek w_h, note that the solution $\nabla^2 w - \lambda w = 0$ is $w = C_1 + C_2\log r$ for $\lambda = 0$ and $w = C_3 I_0(\sqrt{\lambda}\,r) + C_4 K_0(\sqrt{\lambda}\,r)$ where I_0 and K_0 are modified Bessel functions of the first and the second kind of order 0. Seek solution of $(*)$ in the form of $\nabla^2 w = \lambda w$,

$$\lambda^2 - \frac{N_m}{B}\lambda = 0 \rightarrow \lambda_1 = 0,\ \lambda_2 = \frac{N_m}{B} \rightarrow w_h(r) = C_1 + C_2\log r + C_3 I_0\left(\sqrt{\frac{N_m}{B}}\,r\right) + \underbrace{C_4 K_0\left(\sqrt{\frac{N_m}{B}}\,r\right)}_{\sim -\log r \text{ as } r\to 0}$$

With BCs, you'll be able to figure out C_1, C_2, C_3, C_4 and $\mathcal{G} = \mathcal{G}(p)$ or $\mathcal{G}(\delta)$.

It has been shown that as $N_m a^2 \ll B$, $w \rightarrow \frac{pa^4}{64B}\left(1-\frac{r^2}{a}\right)^2$, $\mathcal{G} \rightarrow \frac{p^2 a^4}{128B}$, $\psi \rightarrow -45^\circ$.
(Freund & Suresh, 2003)

Finally, let's discuss the membrane response with $N_m=0$. Once again, start with scalings.



$$\varepsilon \sim \delta^2/a^2 \longrightarrow \varepsilon_0 = \psi(\nu)\,\delta^2/a^2 \quad \text{strain level at the center.}$$

$$p \times \underbrace{a^2\cdot\delta}_{\sim\text{Volume}} \sim Eh\varepsilon^2 \times \underbrace{a^2}_{\sim\text{Area}} \rightarrow p = \xi(\nu)\,Eh\frac{\delta^3}{a^4}$$

$$G \sim \frac{\Pi}{a^2} \sim Eh\varepsilon^2 \rightarrow G = \phi(\nu)\,Eh\frac{\delta^4}{a^4}$$

$$G \sim \frac{pa^2\delta}{a^2} \longrightarrow G = \left[\varphi(\nu)\,\frac{p^4a^4}{Eh}\right]^{1/3}$$

Need to solve the boundary value problem:

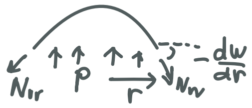
$$N_{rr}\frac{d^2w}{dr^2} + N_{\theta\theta}\frac{1}{r}\frac{dw}{dr} + p = 0 \quad ① \qquad \& \qquad \frac{dN_{rr}}{dr} + \frac{N_{rr}-N_{\theta\theta}}{r} = 0 \quad ②$$

subject to $u(0)=0$, $w'(0)=0$, $w(a)=0$, $u(a)=0 \leftarrow$ Neglected N_m.

· Hencky's solution (1915)

(see NASA Techical Report L-17585)

$$② \rightarrow \frac{d(rN_{rr})}{dr} - N_{\theta\theta} = 0$$



$$①+② \rightarrow N_{rr}\frac{d^2w}{dr^2} + \frac{d(rN_{rr})}{dr}\frac{1}{r}\frac{dw}{dr} = -p \rightarrow \frac{d}{dr}\left(rN_{rr}\frac{dw}{dr}\right) = -pr \rightarrow \boxed{N_{rr}\frac{dw}{dr} = -\frac{1}{2}pr} \quad ③$$

$$\varepsilon_r = \frac{d(r\varepsilon_\theta)}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \rightarrow \frac{1}{2}Eh\left(\frac{dw}{dr}\right)^2 = N_{rr} - \nu N_{\theta\theta} - \frac{d}{dr}\left[r(N_{\theta\theta} - \nu N_{rr})\right]$$

$$② \rightarrow = N_{rr} - \nu\cancel{\frac{d(rN_{rr})}{dr}} - \frac{d}{dr}\left[r\frac{d}{dr}(rN_{rr}) - \nu\cancel{rN_{rr}}\right]$$

$$\rightarrow \boxed{r\frac{d}{dr}\left[\frac{d}{dr}(rN_{rr}) + N_{rr}\right] + \frac{1}{2}Eh\left(\frac{dw}{dr}\right)^2 = 0} \quad ④$$

$$③+④ \rightarrow \boxed{N_{rr}^2\frac{d}{dr}\left[\frac{d}{dr}(rN_{rr}) + N_{rr}\right] + \frac{1}{8}Eh\,p^2 r = 0} \quad ⑤$$

Hencky solved this problem by assuming the following form:

$$N_{rr} = \frac{1}{4}\left(Eh\,p^2a^2\right)^{1/3} \sum_0^{\infty} b_{2n}\left(\frac{r}{a}\right)^{2n}, \quad w = \left(\frac{pa^4}{Eh}\right)^{1/3} \sum_0^{\infty} a_{2n}\left[1-\left(\frac{r}{a}\right)^{2n+2}\right]$$

$$\xrightarrow{②} N_{\theta\theta} = \frac{1}{4}\left(Eh\,p^2a^2\right)^{1/3} \sum_0^{\infty} (2n+1)\, b_{2n}\underbrace{\left(\frac{r}{a}\right)}_{=\rho}^{2n}$$

Plugging these into ⑤ & ③ gives

$$\left(b_0 + b_2\rho^2 + b_4\rho^4 + \cdots\right)^2\left(8b_2\rho + 24b_4\rho^3 + 48\,b_6\rho^5 + \cdots\right) = -8\rho$$

$$\rightarrow \left(b_0^2 b_2 + 1\right)\rho + \left(3b_0^2 b_4 + 2b_0 b_2^2\right)\rho^3 + \left(6b_0^2 b_6 + 6b_0 b_2 b_4 + b_2\left(b_2^2 + 2b_0 b_4\right)\right)\rho^5 + \cdots = 0$$

$$\therefore\ b_2 = -1/b_0^2, \quad b_2 = -2/(3b_0^5), \quad b_6 = -13/(18b_0^8), \cdots, \quad b_{14} = -219241/(63504\, b_0^{20})$$

$$\left(b_0 + b_2\rho^2 + b_4\rho^4 + \cdots\right)\left(a_0 + 2a_2\rho^2 + 3a_4\rho^4 + 4a_6\rho^6 + \cdots\right) = 1$$

$$\rightarrow \left(b_0 a_0 - 1\right) + \left(2b_0 a_2 + b_2 a_0\right)\rho^2 + \left(3b_0 a_4 + 2b_2 a_2 + b_4 a_0\right)\rho^4 + \cdots = 0$$

$$\therefore\ a_0 = 1/b_0, \quad a_2 = 1/(2b_0^4), \quad a_4 = 5/(9b_0^7), \cdots, \quad a_{12} = 17051/(5292\, b_0^{19})$$

Now the only unknown is b_0, to be determined with boundary conditions. Note that $w'(0) = \underbrace{u(0)}_{=\lim_{r\to 0} r\varepsilon_\theta} = 0$, $w(r=a) = 0$ has been satisfied in the assumed form of N_r & w. The "unused" condition comes from $u(r=a) = \frac{N_m(1-\nu)}{Eh}\cdot a = 0$ since $N_m = 0$, i.e.,

$$u(r=a) = r\cdot\frac{1}{Eh}\left(N_{\theta\theta} - \nu N_{rr}\right)\Big|_{r=a} = r\cdot\frac{1}{Eh}\left[\frac{d(rN_{rr})}{dr} - \nu N_{rr}\right]_{r=a} = 0$$

$$\rightarrow (1-\nu)\,b_0 - (3-\nu)\frac{1}{b_0^2} - (5-\nu)\frac{1}{3b_0^5} + \cdots + (15-\nu)\frac{219241}{63504}\frac{1}{b_0^{20}} = 0$$

$\nu = 0.2$, $b_0 = 1.6827$; $\nu = 0.3$, $b_0 = 1.7244$

Jensen (1991): $\delta = \left[\varphi(\nu)\,\frac{p^4a^4}{Eh}\right]^{1/3}$, $\varphi(\nu = 0.5) \approxeq 0.0523$

Komaragiri et at (2005): $p = f(\nu)\,Eh\,\frac{\delta^3}{a^4}$, $f(\nu) = \left(0.7179 - 0.1706\nu - 0.1495\nu^2\right)^{-3}$.

• Simplified Kinematics

Williams (1997), Wan & Lim (1998), Freund & Suresh (2004)
Yue et at.(2012), Dai et at.(2018)

The idea is to assume kinematically admissible deformation fields and then determine the unknown coefficients using the principle of minimum potential energy. For example, a simple, two-parameter form has been used:

$$\underbrace{w(r)=\delta\left(1-\frac{r^2}{a^2}\right)}_{\text{Satisfy } w'(0)=w(a)=0} \quad \& \quad \underbrace{u(r)=u_0\frac{r}{a}\left(1-\frac{r}{a}\right)}_{\text{Satisfy } u(0)=u(a)=0}$$

Then, radial and hoop strain components can be calculated immediately

$$\varepsilon_{rr}=\frac{u_0}{a}\left(1-2\frac{r}{a}\right)+2\frac{\delta^2 r^2}{a^4}, \quad \varepsilon_{\theta\theta}=\frac{u_0}{a}\left(1-\frac{r}{a}\right)$$

The elastic strain energy (per unit area) is

$$U(r)=\frac{Eh}{2(1-\nu^2)}\left(\varepsilon_{rr}^2+2\nu\,\varepsilon_{rr}\varepsilon_{\theta\theta}+\varepsilon_{\theta\theta}^2\right)$$

The total potential energy can be calculated as

$$\Pi(u_0,\delta)=2\pi\int_0^a U(r)\,r\,dr-2\pi p\int_0^a w(r)\,dr$$

The relation between p and δ and u_0 can be obtained by solving

$$\frac{\partial\Pi}{\partial u_0}=0 \quad \& \quad \frac{\partial\Pi}{\partial\delta}=0$$

I will not show the results here since the accuracy given by this method is not satifactory. Dai et al PRL (2018) showed using $u(r)=u_0\frac{r}{a}\left(1-\frac{r^2}{a^2}\right)$ can help slightly. Further improving the accuracy needs more terms in the assumed kinematics but would lose its advantages in simplicity (compared to Hencky).

• Perturbed spherical cap shapes   Dai JAM (2024)

The idea is that the shape of the bulge is not a spherical cap exactly. But it appears quite close. Naturally seek solution around a parabola.

$$w(r)=\begin{cases}\delta\left[1-\left(\frac{r}{a}\right)^{2+\alpha}\right], & |\alpha|\ll 1 \quad \text{Solution I}\\ \delta\left[1-\left(\frac{r}{a}\right)^{2}\right]^{\beta}, & |\beta|\ll 1 \quad \text{Solution II}\\ \frac{\delta}{1+\epsilon}\left[1-\left(\frac{r}{a}\right)^{2}\right]+\frac{\epsilon\delta}{1+\epsilon}\left[1-\left(\frac{r}{a}\right)^{N}\right], & |\epsilon|\ll 1 \quad \text{Solution III}\end{cases}$$

Regarding the in-plane displacement field, instead of assuming a kinematically admissible form, one can directly solve for it based on in-plane equilibrium equation in terms of displacements:

$$\frac{d}{dr}(rN_{rr})-N_{\theta\theta}=0 \rightarrow \frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\underbrace{\frac{1-\nu}{2r}\left(\frac{dw}{dr}\right)^2}_{\text{The nonlinear term now is explicit.}}+\frac{dw}{dr}\frac{d^2w}{dr^2}=0$$

For example, plugging $w(r)$ in solution I, together with $u(0)=u(a)=0$, can give

$$u(r)=\frac{(2+\alpha)(3+2\alpha-\nu)}{8(1+\alpha)}\delta^2\frac{r}{a}\left[1-\left(\frac{r}{a}\right)^{2+2\alpha}\right]$$

Then we can use kinematic relations to calculate ε_{rr}, $\varepsilon_{\theta\theta}$ and $\Pi(\delta,\alpha)$. Finally using principle of minimum potential energy $\frac{\partial\Pi}{\partial\delta}=\frac{\partial\Pi}{\partial\alpha}=0$ as well as $|\alpha|\ll 1$ leads to

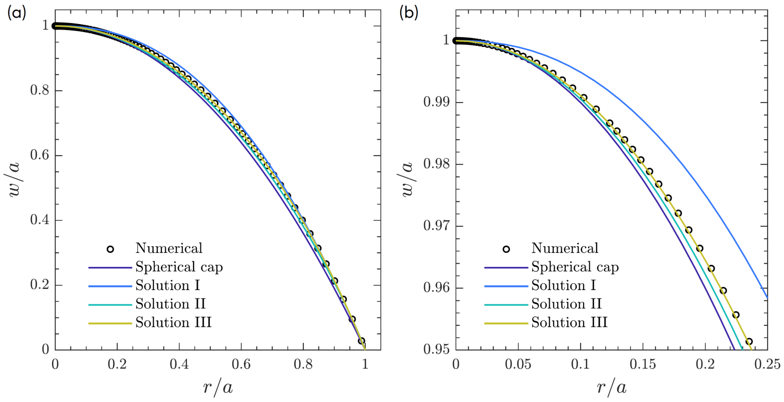
$$\overset{I}{\alpha}(\nu)\approx\frac{\sqrt{1025-742\nu+41\nu^2}-15-3\nu}{50-2\nu},\quad \overset{I}{\delta}=\overset{I}{\delta}(\nu),\quad \overset{I}{\phi}=\overset{I}{\phi}(\nu),\quad \overset{I}{\varphi}=\overset{I}{\varphi}(\nu).$$

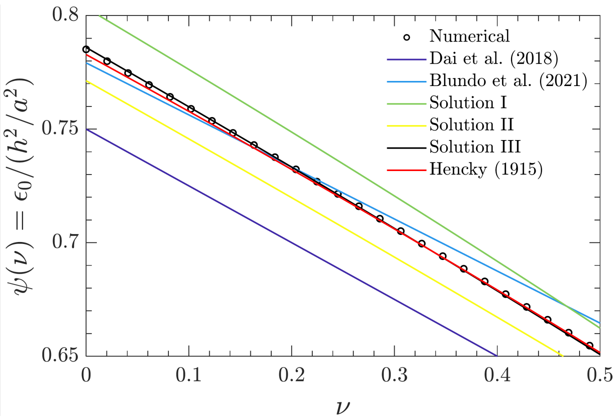
based on solution I

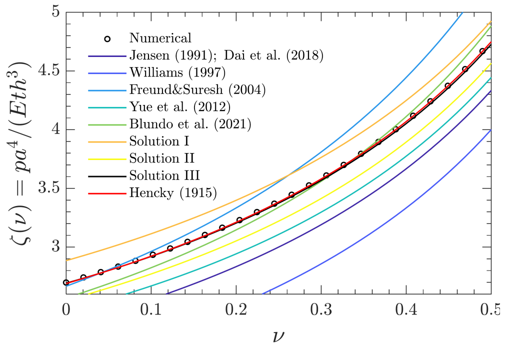
Similarly, these parameters can be calculated by using solution II & III. It it is found Solution III with $N=5$ works particularly well. Specifically,

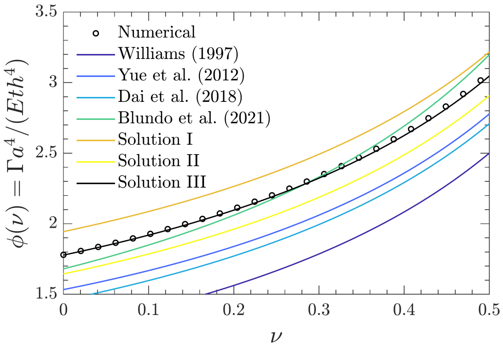
$$\begin{cases} w(r) = \frac{\delta}{1+\epsilon}\left(1-\frac{r^2}{a^2}\right) + \frac{\epsilon\delta}{1+\epsilon}\left(1-\frac{r^5}{a^5}\right), & \epsilon \backsimeq \frac{987-231\nu-7\sqrt{10985+3878\nu-3199\nu^2}}{12(139-67\nu)} \\ \varepsilon_0 = \psi(\nu)\,\delta^2/a^2, & \psi(\nu) = \frac{3-\nu}{4} + \frac{3(1+\nu)}{14}\epsilon + O(\epsilon^2) \\ p = \zeta(\nu)\,Eh\frac{\delta^3}{a^4}, & \zeta(\nu) = \frac{7-\nu}{3(1-\nu)} + \frac{149+13\nu}{63(1-\nu)}\epsilon + O(\epsilon^2) \\ G = \phi(\nu)\,Eh\frac{\delta^4}{a^4}, & \phi(\nu) = \frac{5(7-\nu)}{24(1-\nu)} + \frac{5(53+\nu)}{126(1-\nu}\epsilon + O(\epsilon^2) \\ G = \left[\varphi(\nu)\frac{p^4a^4}{Eh}\right]^{1/3}, & \varphi(\nu) = \frac{375(1-\nu)}{512(7-\nu)} + \frac{625(1-\nu)^2}{448(7-\nu)^2}\epsilon + O(\epsilon^2) \end{cases}$$

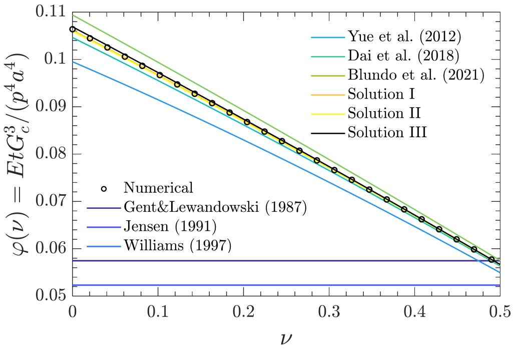
Since $\epsilon \sim 0.1$ for typical ν, this solution reduces the errors within $\epsilon^2 \sim 1\%$.



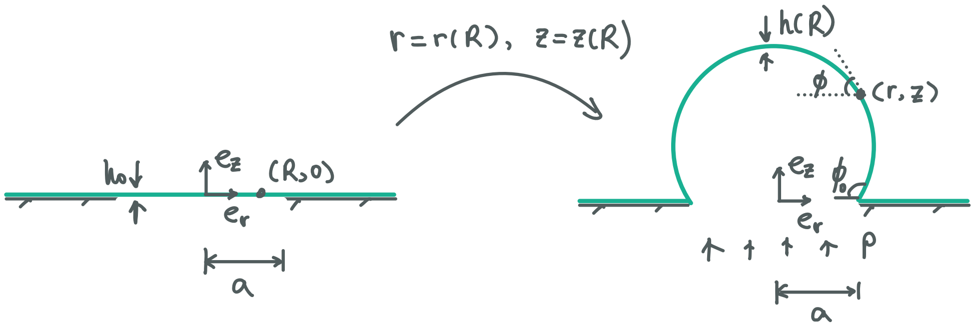








Circular pressurized hyperelastic bulge



λ_r, λ_θ, λ_h denote principal stretches of the membrane along the radial, hoop, and thickness directions.

$$\lambda_r = \sqrt{r'^2 + z'^2}, \quad \lambda_\theta = \frac{r}{R}, \quad \lambda_t = \frac{h \swarrow \text{current thickness}}{h_0 \leftarrow \text{initial thickness.}}$$

We assume the film is incompressible so that $\lambda_r \lambda_\theta \lambda_t \equiv 1$, i.e., $h = h_0 / (\lambda_r \lambda_\theta)$

The volume of the bulge is $V = \int_0^a 2\pi r z dr = \int_0^a 2\pi r r' z dR$

The total potential energy can be written as

$$\Pi = \overset{\text{strain energy}}{U_m} - pV$$

$$= \int_0^a W t_0 2\pi R dR - p \int_0^a 2\pi r r' z dR$$

$$= 2\pi \int_0^a \underbrace{(W t_0 R - p r r' z)}_{= F(r, r', z, z')} dR$$

where $W = W(\lambda_r, \lambda_\theta)$ is the strain energy per unit volume in the undeformed configuration.

Therefore $\Pi = \Pi(r, r', z, z')$. Let's then examine $\delta\Pi$ with $\delta R_0 \neq 0$ since we want to know $\mathcal{G} = -\delta\Pi / \delta(\pi R_0^2)$.

$$\delta \int_0^a F dR = \int_0^a \left(\frac{\partial F}{\partial r} \delta r + \frac{\partial F}{\partial r'} \delta r' + \frac{\partial F}{\partial z} \delta z + \frac{\partial F}{\partial z'} \delta z' \right) dR + F\Big|_a \delta a$$

Note that $\delta \int_0^a \frac{\partial F}{\partial r'} \delta r' dR = \frac{\partial F}{\partial r'} \delta r \Big|_a - \frac{\partial F}{\partial r'} \delta r \Big|_0 - \int_0^a \frac{d}{dR}\left(\frac{\partial F}{\partial r'}\right) \delta r dR$

$$\delta \int_0^a \frac{\partial F}{\partial z'}\delta z' dR = \frac{\partial F}{\partial z'}\delta z\Big|_a - \frac{\partial F}{\partial z'}\delta z\Big|_0 - \int_0^a \frac{d}{dR}\left(\frac{\partial F}{\partial z'}\right)\delta z \, dR$$

$$\rightarrow \delta\Pi = \int_0^a \Big[\underbrace{\left(\frac{\partial F}{\partial r} - \frac{d}{dR}\frac{\partial F}{\partial r'}\right)}_{=0}\delta r + \underbrace{\left(\frac{\partial F}{\partial z} - \frac{d}{dR}\frac{\partial F}{\partial z'}\right)}_{=0}\delta z\Big] dR$$

$$+ \left(\frac{\partial F}{\partial r'}\delta r\Big|_a + \frac{\partial F}{\partial z'}\delta z\Big|_a + F\Big|_a \delta a - \frac{\partial F}{\partial r'}\delta r\Big|_0 - \frac{\partial F}{\partial z'}\delta z\Big|_0\right) \xleftarrow{} \text{B.T.s.} = 0$$

$$= 0$$

Let's first examine the two Lagrangians

$$\frac{\partial F}{\partial r} - \left(\frac{\partial F}{\partial r'}\right)' = \frac{\partial W}{\partial r} t_0 R - p r' z - \left(R\frac{\partial W}{\partial r'}\right)' t_0 + p(r'z + rz')$$

$$= t_0\left[R\frac{\partial W}{\partial r} - \left(R\frac{\partial W}{\partial r'}\right)'\right] + p r z' = 0 \qquad ①$$

$$\frac{\partial F}{\partial z} - \left(\frac{\partial F}{\partial z'}\right)' = t_0\left[R\frac{\partial W}{\partial z} - \left(R\frac{\partial W}{\partial z'}\right)'\right] - p r r' = 0 \qquad ②$$

W is a function of $\lambda_r, \lambda_\theta$. Need to establish the relation between $\frac{\partial W}{\partial r}$ and $\frac{\partial W}{\partial \lambda_r}$ etc.

$$\frac{\partial W}{\partial r} = \frac{\partial W}{\partial \lambda_r}\cancelto{0}{\frac{\partial \lambda_r}{\partial r}} + \frac{\partial W}{\partial \lambda_\theta}\cdot\frac{\partial \lambda_\theta}{\partial r} = \frac{1}{R}\frac{\partial W}{\partial \lambda_\theta}$$

$$\frac{\partial W}{\partial r'} = \frac{\partial W}{\partial \lambda_r}\cdot\frac{\partial \lambda_r}{\partial r'} + \frac{\partial W}{\partial \lambda_\theta}\cancelto{0}{\frac{\partial \lambda_\theta}{\partial r'}} = \frac{r'}{\sqrt{r'^2+z'^2}}\cdot\frac{\partial W}{\partial r'} = \frac{r'}{\lambda_r}\frac{\partial W}{\partial \lambda_r}$$

$$\frac{\partial W}{\partial z} = 0, \quad \frac{\partial W}{\partial z'} = \frac{z'}{\lambda_r}\frac{\partial W}{\partial \lambda_r}$$

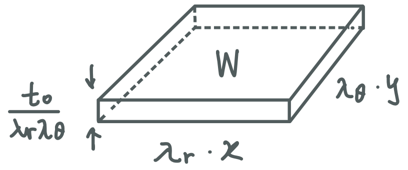
With these, we can rewrite ① and ② as

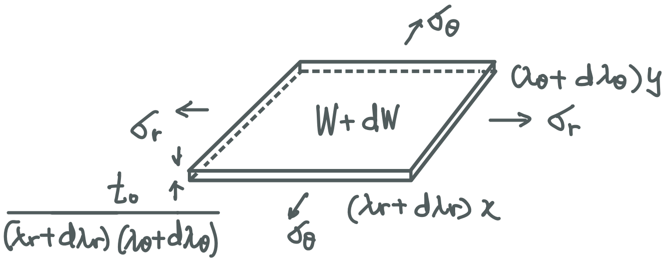
$$p = -\frac{t_0}{rz'}\left[\frac{\partial W}{\partial \lambda_\theta} - \frac{r'}{\lambda_r}\frac{\partial W}{\partial \lambda_r} - R\left(\frac{r'}{\lambda_r}\right)'\frac{\partial W}{\partial \lambda_r} - R\frac{r'}{\lambda_r}\left(\frac{\partial W}{\partial \lambda_r}\right)'\right]$$

$$③$$

$$p = \frac{t_0}{rr'}\left[-\frac{z'}{\lambda_r}\frac{\partial W}{\partial \lambda_r} - R\left(\frac{z'}{\lambda_r}\right)'\frac{\partial W}{\partial \lambda_r} - R\left(\frac{z'}{\lambda_r}\right)\left(\frac{\partial W}{\partial \lambda_r}\right)'\right]$$

What is the physical picture of $\frac{\partial W}{\partial \lambda_r}, \frac{\partial W}{\partial \lambda_\theta}$? We learnt $\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}$ so expect $\frac{\partial W}{\partial \lambda_r}$ and $\frac{\partial W}{\partial \lambda_\theta}$ will give rise to something like stress.





Stored energy: $\underbrace{W(\lambda_r, \lambda_\theta)}\, xy\, t_0$

defined in the undeformed configuration but it is the same in the deformed configuration when the material is incompressible:

$$\tilde{W}(\lambda_r, \lambda_\theta) \cdot \lambda_r x \cdot \lambda_\theta y \cdot \frac{t_0}{\lambda_r \lambda_\theta} = \tilde{W}\, xy\, t_0$$

Stored energy: $W(\lambda_r + d\lambda_r, \lambda_\theta + d\lambda_\theta)\, xy\, t_0$

$$= \left[W(\lambda_r, \lambda_\theta) + \frac{\partial W}{\partial \lambda_r} d\lambda_r + \frac{\partial W}{\partial \lambda_\theta} d\lambda_\theta \right] xy\, t_0$$

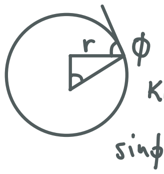
$$\overbrace{\underbrace{\sigma_r \cdot \lambda_\theta y \cdot \frac{t_0}{\lambda_r \lambda_\theta}}_{\text{Net force}} \cdot \underbrace{x \cdot d\lambda_r}_{\text{displc.}} + \sigma_\theta\, \lambda_r x \cdot \frac{t_0}{\lambda_r \lambda_\theta} \cdot y\, d\lambda_\theta}^{\text{External work}} = \overbrace{\left(\frac{\partial W}{\partial \lambda_r} d\lambda_r + \frac{\partial W}{\partial \lambda_\theta} d\lambda_\theta \right) xy\, t_0}^{\text{Change in stored energy}}$$

$$\rightarrow \left(\frac{\sigma_r}{\lambda_r} - \frac{\partial W}{\partial \lambda_r} \right) d\lambda_r + \left(\frac{\sigma_\theta}{\lambda_\theta} - \frac{\partial W}{\partial \lambda_\theta} \right) d\lambda_\theta = 0$$

$d\lambda_r$, $d\lambda_\theta$ are arbitary so that $\sigma_r = \lambda_r \frac{\partial W}{\partial \lambda_r}$, $\sigma_\theta = \lambda_\theta \frac{\partial W}{\partial \lambda_\theta}$. We are interested in stress resultants: $N_r = \sigma_r \cdot t = \frac{t_0}{\lambda_\theta} \frac{\partial W}{\partial \lambda_r}$, $N_\theta = \sigma_\theta t = \frac{t_0}{\lambda_r} \frac{\partial W}{\partial \lambda_\theta}$. Now rewrite ③ in terms of N_r, N_θ

$$p = -\frac{1}{rz'} \left[\lambda_r N_\theta - \frac{r'}{\lambda_r} \lambda_\theta N_r - R\left(\frac{r'}{\lambda_r}\right)' \cdot \lambda_\theta N_r - R\frac{r'}{\lambda_r} (\lambda_\theta N_r)' \right]$$

$$= -\frac{(r'^2 + z'^2)^{1/2}}{rz'} N_\theta + \frac{r'^4 + r'^2 z'^2 + r r'' z'^2 - r r' z z'}{rz'(r'^2 + z'^2)^{3/2}} N_r + \frac{r'}{z'(r'^2 + z'^2)^{1/2}} \frac{dN_r}{dR} \quad ④$$

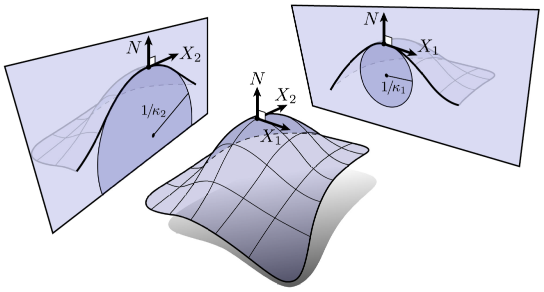


$\kappa_\theta = -\frac{\sin\phi}{r}$

$\sin\phi = -\frac{z'}{\lambda_r}$

$$p = \frac{1}{rr'} \left[-\frac{z'}{\lambda_r} \lambda_\theta N_r - R\left(\frac{z'}{\lambda_r}\right)' \lambda_\theta N_r - R\frac{z'}{\lambda_r} (\lambda_\theta N_r)' \right]$$

$$= -\frac{z'(r'^2 + z'^2 - r r'') + r r' z''}{r(r'^2 + z'^2)^{3/2}} N_r - \frac{z'}{r'(r'^2 + z'^2)^{1/2}} \frac{dN_r}{dR} \quad ⑤$$



$$④-⑤ \rightarrow \boxed{\frac{dN_r}{dR} + \frac{r'(N_r - N_\theta)}{r} = 0}$$

$$④+⑤ \rightarrow \boxed{\underbrace{\frac{r'z'' - z'r''}{(r'^2 + z'^2)^{3/2}}}_{K_r} N_r + \underbrace{\frac{z'}{r(r'^2 + z'^2)^{1/2}}}_{K_\theta} N_\theta = p}$$

$$\xrightarrow[r \to R,\ r' \to 1,\ |z'^2| \ll 1]{\text{Moderate rotation}} \begin{cases} \frac{dN_r}{dR} + \frac{N_r - N_\theta}{R} = 0 \\ N_{rr} z'' + N_{\theta\theta} \frac{z''}{r} = 0 \end{cases}$$

(Foppel membrane)

Material law

Having given the equilibrium equations, we need to specify a material law to proceed. There are various types of material laws - we consider two of commonly used for soft materials.

$$\text{Neo-Hookean model}: \quad W = \overset{\swarrow \text{ shear modulus of the membrane}}{\frac{\mu}{2}} \Big(\underbrace{\lambda_r^2 + \lambda_\theta^2 + \frac{1}{\lambda_r^2 \lambda_\theta^2}}_{=I_1} - 3 \Big)$$

$$\text{Gent model}: \quad W = -\frac{\mu}{2} J_m \log \Big(1 - \frac{I_1 - 3}{J_m} \Big)$$

material constant ↗ (J_m)

first invariant of Cauchy-Green tensor ↙ (I_1)

i.e. $I_1 = \mathrm{tr}(\underset{\approx}{F}\underset{\approx}{F}^T)$

$= \lambda_r^2 + \lambda_\theta^2 + \frac{1}{\lambda_r^2 \lambda_\theta^2}$

Note that as $J_m \gg 1$, Gent model behaves as $\frac{\mu}{2}(I_1 - 3 + O(J_m^{-1}))$. So we will try Gent model with a range of J_m

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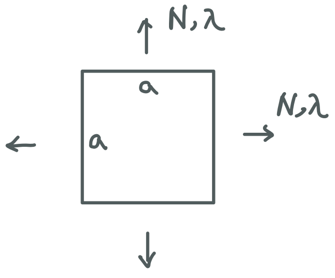
Aside.

Consider equibiaxial tension $\lambda_r = \lambda_\theta = \lambda$, $\lambda_h = 1/\lambda^2$

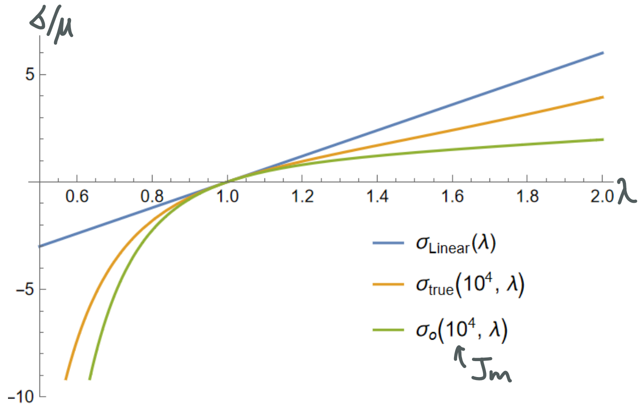


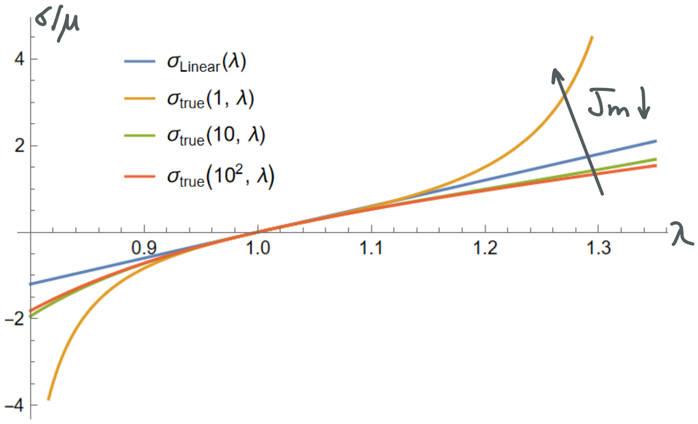
$$\underset{\text{true stress}}{\sigma} = \lambda_r \frac{\partial W}{\partial \lambda_r}\Big|_{\lambda_r = \lambda_\theta = \lambda} = \frac{1}{2}\lambda \frac{\partial W}{\partial \lambda} = \mu J_m \frac{\lambda^6 - 1}{-2\lambda^6 + (3 + J_m)\lambda^4 - 1}$$

$$F = \sigma t a = \frac{t_0}{\lambda^2} a_0 \lambda \sigma$$

$$\rightarrow \underset{\text{nominal stress}}{\sigma_0} = \frac{F}{a_0 t_0} = \sigma/\lambda = 2\mu J_m \frac{\lambda^6 - 1}{-2\lambda^7 + (3 + J_m)\lambda^5 - \lambda}$$

Linear Hooke's law $\sigma = \sigma_0 = \frac{E}{1-\nu}\varepsilon = 6\mu(\lambda - 1)$ since $\mu = \frac{E}{2(1+\nu)} = \frac{E}{3}$ and $\nu = 0.5$





Using Gent material model, we obtain

$$N_r = \mu t_0 J_m \frac{\lambda_r^4 \lambda_\theta^2 - 1}{\lambda_r^3 \lambda_\theta^3 (J_m + 3 - \lambda_r^2 - \lambda_\theta^2) - \lambda_r \lambda_\theta}$$

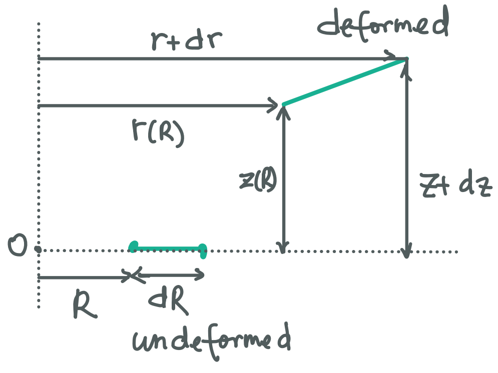
$$N_\theta = \mu t_0 J_m \frac{\lambda_r^2 \lambda_\theta^4 - 1}{\lambda_r^3 \lambda_\theta^3 (J_m + 3 - \lambda_r^2 - \lambda_\theta^2) - \lambda_r \lambda_\theta}$$

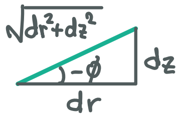
Numerics

Let's first solve the system with some natural boundary conditions

$$z'(0)=0, \quad z(a)=0, \quad r(0)=\lim_{R\to 0}(R+u(R))=0, \quad r(a)=\lim_{R\to a}(R+u(a))=a \quad \text{or } \lambda_\theta(a)=1$$

You'll find bvp solvers not quite efficient due to a good deal of nonlinearities. May try to solve a ivp problem using shooting method. The idea is to replace 2 second order coupled ODEs regarding r'', z'' with 4 first order ODEs. There are many options while we take $\lambda_\theta=\lambda_\theta(R)$, $\lambda_z=\lambda_z(R)$, $z=z(R)$, $\phi=\phi(R)$ here





→

$$\boxed{\cos\phi = \frac{r'}{\lambda_r}, \quad \sin\phi = -\frac{z'}{\lambda_r} \checkmark \qquad K_r = -\frac{\phi'}{\lambda_r}, \quad K_\theta = -\frac{\sin\phi}{r}}$$

$- d\phi/ds = -\phi' dR/ds = -\phi'/\lambda_r$

Need to know relations between λ_r', λ_θ', ϕ' and $\lambda_r, \lambda_\theta, \phi, z$

$$\lambda_\theta' = \frac{r'}{R} - \frac{r}{R^2} = \frac{\lambda_r \cos\phi}{R} - \frac{\lambda_\theta}{R} \checkmark$$

$$\frac{dN_r}{dR} + \frac{r'(N_r - N_\theta)}{r} = 0 \rightarrow \frac{dN_r}{d\lambda_\theta} \cdot \lambda_\theta' + \frac{dN_r}{d\lambda_r} \cdot \lambda_r' + \frac{r'(N_r - N_\theta)}{r} = 0$$

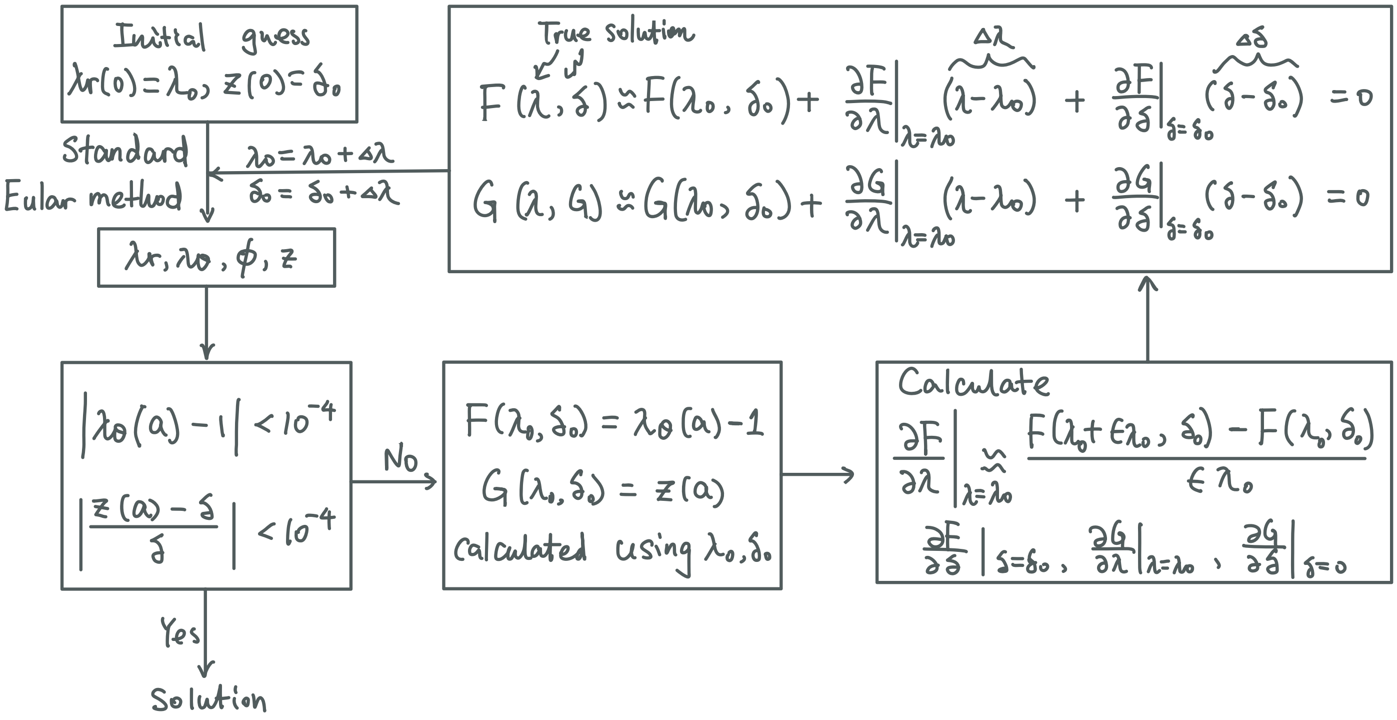
$$\rightarrow \lambda_r' = \left[-\frac{\lambda_r \cos\phi (N_r - N_\theta)}{\lambda_\theta R} - \frac{\lambda_r \cos\phi - \lambda_\theta}{R} \frac{dN_r}{d\lambda_\theta} \right] \Big/ \frac{dN_r}{d\lambda_r} \checkmark$$

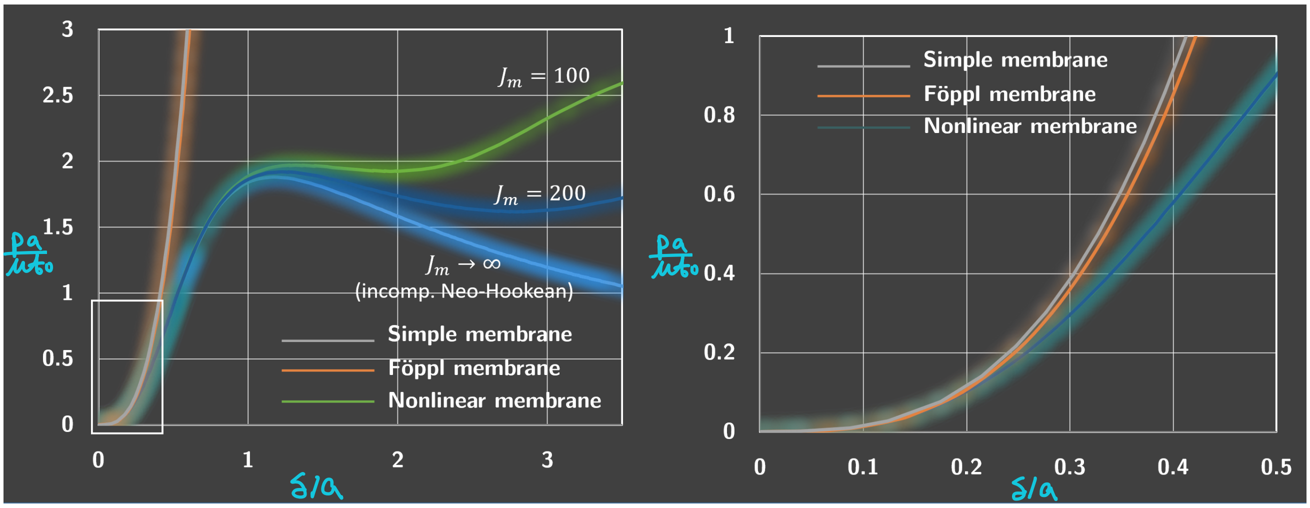
$$N_r K_r + N_\theta K_\theta = -p \rightarrow -\frac{\phi'}{\lambda_r} N_r - \frac{\sin\phi}{r} N_\theta = -p \rightarrow \phi' = \left(p - \frac{\sin\phi}{\lambda_\theta R} N_\theta\right) \lambda_r / N_r \checkmark$$

These equations complete the 4 ODEs : $\frac{dy_i}{dR} = f_i(y_i, R)$, which can be solved with initial conditions $y_i(0)$, given other parameters including μ, a, J_m, p:

$$\lambda_r(0) = \lambda_\theta(0) = \lambda \ , \quad \phi(0)=0, \quad z(0)=\delta$$

However, λ and δ are not known a priori – the value of them should ensure that $\lambda_\theta = 1$ and $z = 0$ at $R = a$.





- Simple membrane : $N\nabla^2 w = -P$
- Föppl membrane : $N_{rr}K_{rr} + N_{\theta\theta}K_{\theta\theta} = -P$, $\frac{d(RN_r)}{dR} - N_\theta = 0$, $K_{rr}, K_{\theta\theta}$ linearied.

$\Big\}$ Good only when $\left(\frac{\delta}{a}\right)^2 \ll 1$

- Nonlinear membran : ① Snap through and stiffening for small J_m.

Energy release rate

The membrane-substrate interface toughness (say $\mathcal{G}_c$) is finite. Then interested in at which pressure the interface breaks, i.e., $\mathcal{G}(p) = \mathcal{G}_c$. How to calculate this? $\mathcal{G} = -\frac{\partial \Pi}{\partial(\pi a^2)}$

One immediate way is to compute Π_1 at given a and p and Π_2 at $a+\epsilon a$ with $0<\epsilon<<1$. Then $\mathcal{G}(p) = -\lim\limits_{\epsilon\to 0} \frac{\Pi_2-\Pi_1}{2\pi a(\epsilon a)} = \lim\limits_{\epsilon\to 0} \frac{\Pi_1-\Pi_2}{2\pi a^2 \epsilon}$. The other way is to fing $\delta\Pi/(2\pi a\,\delta a)$ via variational analysis. Now revisit the boundary terms on Page 108:

$$\frac{\partial F}{\partial r'}\delta r\Big|_a + \frac{\partial F}{\partial z'}\delta z\Big|_a + F\Big|_a \delta a - \frac{\partial F}{\partial r'}\delta r\Big|_0 - \frac{\partial F}{\partial z'}\delta z\Big|_0 = 0$$

where $F(r,r',z,z') = W(r,r',z')\,t_0 R - p\,r\,r' z$.

P108 P109

$$\frac{\partial F}{\partial r'}\delta r = \left(\frac{\partial W}{\partial r'} t_0 R - prz\right)\delta r = \left(\frac{r'}{\lambda_r}\frac{\partial W}{\partial \lambda_r} t_0 R - prz\right)\delta r = \left(\frac{r' N_r \lambda_\theta}{\lambda_r} R - prz\right)\delta r$$

$$\delta r|_0 = \delta r(0) = 0 \quad \text{but} \quad \delta r|_a = \delta r(a) - r'\Big|_a \delta a = (1 - r'|_a)\delta a$$

$$\rightarrow \frac{\partial F}{\partial r'}\delta r\Big|_0 = 0, \quad \frac{\partial F}{\partial r'}\delta r\Big|_a = \left[\left(\frac{r' N_r \lambda_\theta}{\lambda_r} R - pr\cancelto{0}{z}\right)(1-r')\right]_{R=a}\delta a \quad \leftarrow r' = \lambda_r \cos\phi$$

$$= N_r\Big|_a \cos\phi_0 \cdot a\,(1 - \cos\phi_0 \lambda_r\Big|_a)\,\delta a$$

$$\frac{\delta F}{\partial z'}\delta z = \left(\frac{\partial W}{\partial z'} t_0 R - prr'\right)\delta z = \left(\frac{z'}{\lambda_r}\frac{\partial W}{\partial \lambda_r} t_0 R - pr\lambda_r\cos\phi\right)\delta z = \left(\frac{z' N_r \lambda_\theta}{\lambda_r} R - pr\lambda_r\cos\phi\right)\delta z$$

$$\delta z|_0 = \delta z(0), \quad \delta z|_a = \delta \cancelto{0}{z(a)} - z'|_a\,\delta a$$

$$\rightarrow \frac{\partial F}{\partial z'}\delta z\Big|_0 = 0\ \delta z(0) = 0, \quad \frac{\partial F}{\partial z'}\delta z\Big|_a = -\left(\frac{z' N_r \lambda_\theta}{\lambda_r} R - pr\lambda_r\cos\phi\right)\cdot z'\Big|_{R=a}\delta a \quad \leftarrow z' = -\lambda_r \sin\phi$$

$$= -\left(N_r|_a \sin\phi_0 \cdot a + pa\lambda_r\Big|_a \cos\phi\Big|_0\right)\lambda_r\Big|_a \sin\phi_0\ \delta a$$

$$F|_a\,\delta a = \left(W|_a t_0 a + p\cdot a\cdot \lambda_r^2|_a \cos\phi_0 \sin\phi\right)\delta a$$

$$\rightarrow \left(N_r\Big|_a \cos\phi_0 a - (N_r\lambda_r)|_a \cos^2\phi_0\, a - (N_r\lambda_r)|_a \sin^2\phi_0\, a - pa\,\cancel{\lambda_r^2|_a \cos\phi_0} \sin\phi_0 + W|_a t_0 a + pa\,\cancel{\lambda_r^2|_a \cos\phi_0} \sin\phi_0\right)\delta a$$

$$\rightarrow \mathcal{G} = -\frac{\partial \Pi}{\partial(\pi a^2)} = -\frac{\partial \Pi}{\underbrace{2\pi a}_{\text{cancelled out when using } \delta F}\partial a} = \left(N_r \lambda_r - N_r \cos\phi - t_0 W\right)_{R=a}$$

At small stretches, $\lambda_r \rightarrow 1 + \varepsilon_r$, $W \rightarrow \frac{1}{2} N_r \varepsilon_r + \frac{1}{2} \cancel{N_\theta \varepsilon_\theta}^{\text{at } R=a}$

$$\mathcal{G} = \left[N_r(1-\cos\phi) + \frac{1}{2} N_r \varepsilon_r\right]_{R=a} ,$$

which returns to Kendall's peeling angle obtained using linear elasticity.