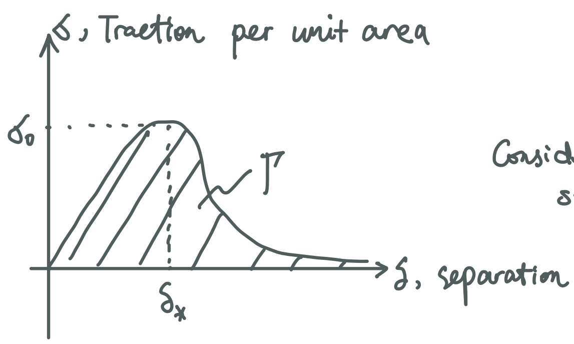
The Dugdale - Barenblatt model

In a linear elastic material, K is related to the coefficient on the singular $1/\sqrt{r}$ stress terms, arising near a crack tip. However, we have noted that real materials cannot support infinite stresses. Why can K still be used as a fracture criterion if it doesn't represent the reality?

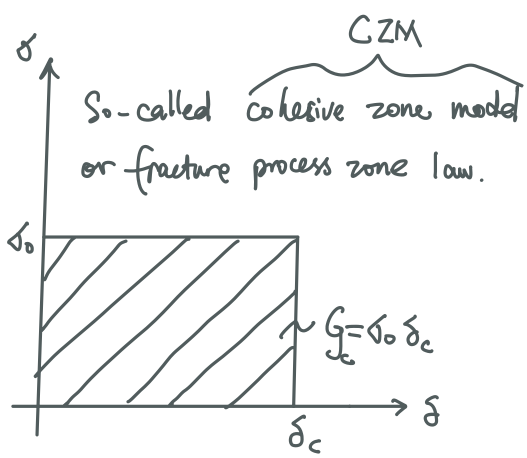
Consider the following model for nonlinear behavior near the crack tip. We will only analyze the nonlinear behavior on the plane ahead of the crack tip and will assume that σ_{yy} is limited to a peak value of σ_0. Later, we will show σ_{yy} can take any form under SSY conditions.

Recall the molecular view of fracture

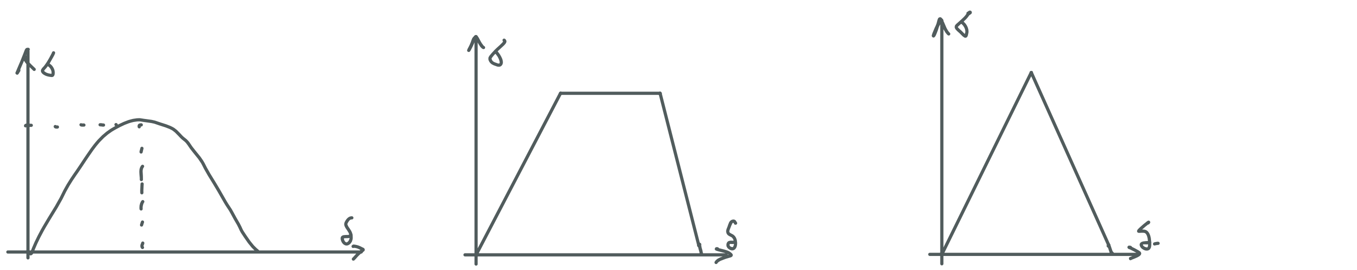


$\Rightarrow$

Consider a simplified situation

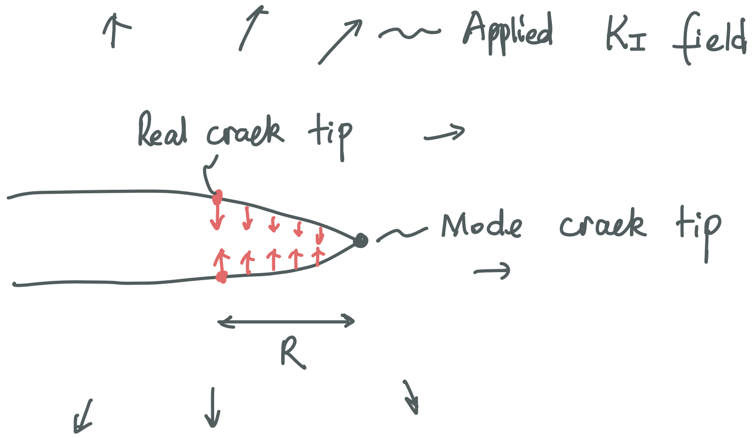


CZM can be used to repesent a vast number of fracture mechanisms including plastic tearing, atomic de-cohesion, fiber pull-out in composite fracture.

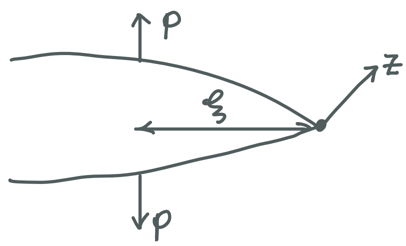


There are also a number of CZMs, but let's focus on constant σ_0 at this moment!

The crack model looks like:



We can set our origin at the real crack tip or the mode crack tip. Let us choose the latter since we've already have the solution in HW3:



$$Z_I = \frac{P\sqrt{\xi}}{\pi\sqrt{z}\,(z+\xi)}$$

$$K_I^P = P\sqrt{\frac{2}{\pi\xi}}$$

At the mode crack tip, we have

$$K_{tip} = \underbrace{K_{App}}_{\text{Applied K field}} + \underbrace{K_{CZM}}_{\text{Due to cohesive tractions}}$$

$$= K_I + \int_0^R \underbrace{-\sigma_0 d\xi}_{P} \sqrt{\frac{2}{\pi\xi}}$$

What is K_{tip}? If $K_{tip} \neq 0$, then σ_{yy} ahead of the model crack tip will go to ∞. This is prohibited by the CZM (it avoids singularity by "tuning" R).

$$\to \quad K_I - \sigma_0\sqrt{\frac{2}{\pi}}\int_0^R \xi^{-1/2}\, d\xi = K_I - \sqrt{\frac{2}{\pi}}\,\sigma_0\, 2\,\xi^{1/2}\Big|_0^R = 0$$

$$\to \quad K_I = \sqrt{\frac{8}{\pi}}\,\sigma_0 R^{1/2} \quad \text{or} \quad \boxed{R = \frac{\pi}{8}\left(\frac{K_I}{\sigma_0}\right)^2}$$ Size of Dugdale plastic zone

Before we say fracture occurs when $K_I = K_{Ic}$ or $\frac{K_I^2}{E'} = \mathcal{G}_c$. Now, we need a new criterion. This criterion is obvious from the point of view of physics

$$\delta(z=-R) = \delta_c$$

To compute the crack opening displacement, we need to calculate Z_I & $\hat{Z}_I = \int Z_I dz$, since $2\mu u_y = \frac{1}{2}(\kappa+1)\,\mathrm{Im}\,\hat{Z}_I - y\,\mathrm{Re}\,Z_I$.

$$Z_I = \frac{K_I}{\sqrt{2\pi z}} - \int_0^R \frac{\sigma_0\sqrt{\xi}}{\pi\sqrt{z}(z-\xi)}\,d\xi$$

$$= \frac{K_I}{\sqrt{2\pi z}} - \frac{\sigma_0}{\pi\sqrt{z}}\left[2\sqrt{R} - 2\sqrt{z}\arctan\sqrt{\frac{R}{z}}\right]$$

$$= \cancel{\frac{K_I}{\sqrt{2\pi z}} - \frac{2\sigma_0}{\pi\sqrt{z}}\sqrt{\frac{\pi}{8}}\frac{K_I}{\sigma_0}} \to 0 + \boxed{\frac{2\sigma_0}{\pi}\arctan\sqrt{\frac{R}{z}}}$$

$$\hat{Z}_I = \frac{2\sigma_0}{\pi}\left(\sqrt{Rz} + z\arctan\sqrt{\frac{R}{z}} - R\arctan\sqrt{\frac{z}{R}}\right)$$

Need a bit thougt to compute arctan x

$$\left.\begin{aligned} \sin\theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = -i\sinh i\theta \\ \cos\theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \cosh i\theta \\ \tan\theta &= -i\tanh i\theta \end{aligned}\right\} \quad \begin{aligned} x &= \tan\theta = -i\tanh i\theta \\ \arctan x &= \theta = \frac{1}{i}\tanh^{-1} ix = \frac{1}{2i}\ln\left(\frac{1+ix}{1-ix}\right) \end{aligned}$$

On the crack surfaces, $z = re^{\pm i\pi}$, $y = 0^\pm$:

$$\hat{Z}_I = \frac{2\sigma_0}{\pi}\left[\sqrt{Rr}\,e^{\pm i\frac{\pi}{2}} + re^{\pm i\pi}\frac{1}{2i}\ln\left(\frac{1+i\sqrt{\frac{R}{r}}e^{\mp i\frac{\pi}{2}}}{1-i\sqrt{\frac{R}{r}}e^{\mp i\frac{\pi}{2}}}\right) - R\frac{1}{2i}\ln\left(\frac{1+i\sqrt{\frac{r}{R}}e^{\pm i\frac{\pi}{2}}}{1-i\sqrt{\frac{r}{R}}e^{\pm i\frac{\pi}{2}}}\right)\right]$$

$$= \frac{2\sigma_0}{\pi}\left[\pm i\sqrt{Rr} + i\frac{r}{2}\ln\left(\frac{1\pm\sqrt{R/r}}{1\mp\sqrt{R/r}}\right) + i\frac{R}{2}\ln\left(\frac{1\mp\sqrt{r/R}}{1\pm\sqrt{r/R}}\right)\right]$$

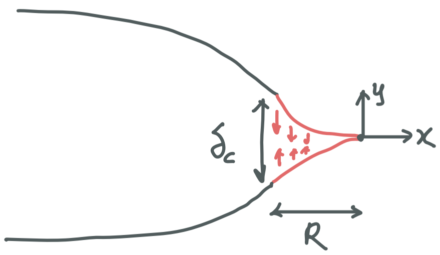
Note that $Re\left[\ln\left(\frac{1+x}{1-x}\right)\right] = Re\left[\ln\left(\left|\frac{1+x}{1-x}\right|e^{in\pi}\right)\right] = \ln\left|\frac{1+x}{1-x}\right|$, $Re\left[\ln\left(\frac{1-x}{1+x}\right)\right] = -\ln\left|\frac{1+x}{1-x}\right|$

$$Im[\hat{Z}_I] = \frac{2\sigma_0}{\pi}\left(\pm\sqrt{Rr} \pm \frac{r}{2}\underbrace{\ln\left|\frac{1+\sqrt{R/r}}{1-\sqrt{R/r}}\right|}_{=\left|\frac{1+\sqrt{r/R}}{1-\sqrt{r/R}}\right|} \mp \frac{R}{2}\ln\left|\frac{1+\sqrt{r/R}}{1-\sqrt{r/R}}\right|\right)$$

$$= \frac{2\sigma_0}{\pi}\left(\pm\sqrt{Rr} \pm \frac{r-R}{2}\ln\left|\frac{1+\sqrt{r/R}}{1-\sqrt{r/R}}\right|\right)$$

$$\rightarrow u_y(r, \theta = \pm\pi) = \frac{\kappa+1}{4\mu}\cdot\frac{2\sigma_0}{\pi}\cdot\left(\pm\sqrt{Rr} \pm \frac{r-R}{2}\ln\left|\frac{1+\sqrt{r/R}}{1-\sqrt{r/R}}\right|\right)$$

$$\frac{u_y}{R}(r, \theta = \pm\pi) = \pm\frac{4\sigma_0}{\pi E'}\left[\underbrace{\sqrt{\frac{r}{R}} - \frac{1}{2}\left(1-\frac{r}{R}\right)\ln\left|\frac{1+\sqrt{r/R}}{1-\sqrt{r/R}}\right|}_{= \begin{cases} \frac{2}{3}\left(\frac{r}{R}\right)^{3/2} & \text{as } \frac{r}{R}\to 0 \\ 1 & \text{as } \frac{r}{R}\to 1 \end{cases}}\right]$$



$$COD(r=R) = u_y(R,\pi) - u_y(R,-\pi) = \frac{8\sigma_0}{\pi E'}R$$

Propogation $\rightarrow$ $\frac{8\sigma_0}{\pi E'}R = \delta_c$

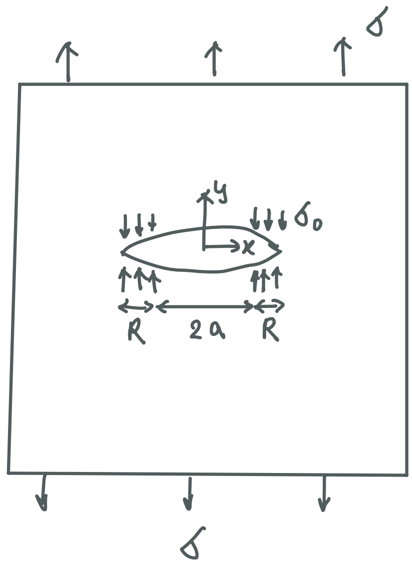
$$\frac{8\sigma_0}{\pi E'}\cdot\frac{\pi}{8}\left(\frac{K_I}{\sigma_0}\right)^2 = \delta_c$$

$$\frac{K_I^2}{E'} = \sigma_0\delta_c \iff \mathcal{G} = \mathcal{G}_c$$

So even though we have no singularity ahead of the crack tip, we still get crack propagation when $\mathcal{G} = \mathcal{G}_c$ or equivalently when $K_I = K_{Ic} = \sqrt{\mathcal{G}_c E'} = \sqrt{\sigma_0\delta_c E'}$.

In the semi-infinite crack problem, we have SSY conditions immediately. Let's move on to the center crack problem in which we have a length scale – initial crack length.

You may use HW3 (problem 1) to show that



$$R = a\underbrace{\left[\sec\left(\frac{\pi}{2}\frac{\sigma}{\sigma_0}\right)-1\right]}_{=\frac{\pi^2}{8}\left(\frac{\sigma}{\sigma_0}\right)^2 + O\left(\frac{\sigma^4}{\sigma_0^4}\right)}$$

$$COD(x=\pm a) = \frac{8}{\pi}\frac{\sigma_0 a}{E'}\ln\left[\sec\left(\frac{\pi}{2}\frac{\sigma}{\sigma_0}\right)\right]$$

Crack propagation $\rightarrow$ $COD(x=\pm a) = \delta_c$

$$\underbrace{\sigma_0\delta_c}_{\text{Will show this is } J \text{ under SSY}} = \frac{8}{\pi}\frac{\sigma_0^2 a}{E'}\underbrace{\ln\left[\sec\left(\frac{\pi}{2}\frac{\sigma}{\sigma_0}\right)\right]}_{=\frac{\pi^2}{8}\left(\frac{\sigma}{\sigma_0}\right)^2+\frac{\pi^4}{192}\left(\frac{\sigma}{\sigma_0}\right)^4+\cdots} = \underbrace{\frac{\pi a\sigma^2}{E'}\left[1+\frac{\pi^2}{24}\left(\frac{\sigma}{\sigma_0}\right)^2+\cdots\right]}_{\text{Will show this is } J \text{ under LSY.}}$$

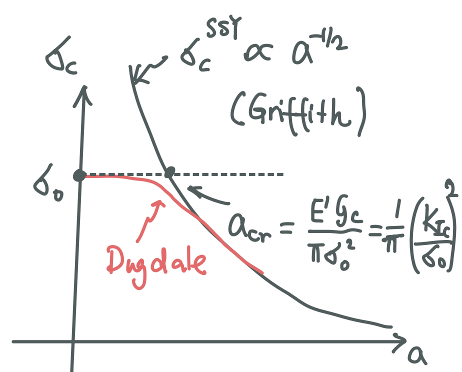
SSY: in the limit as $R \ll a$ (i.e., $\sigma \ll \sigma_0$)

$$K_I = \sigma\sqrt{\pi(a+R)} \rightarrow \sigma\sqrt{\pi a}, \quad \mathcal{G}^{SSY} = \frac{K_I^2}{E'} = \frac{\pi a\sigma^2}{E'}, \quad \sigma_c^{SSY} = \left(\frac{E'\mathcal{G}_c}{\pi a}\right)^{1/2}$$

LSY: R/a is not too small, the apparent strength is

$$\sigma_c^{LSY} = \frac{2\sigma_0}{\pi}\sec^{-1}\left[\exp\left(\frac{\pi E'\mathcal{G}_c}{8\sigma_0^2 a}\right)\right] = \frac{2\sigma_0}{\pi}\sec^{-1}\left[\exp\left(\frac{\pi^2}{8}\frac{{\sigma_c^{SSY}}^2}{\sigma_0^2}\right)\right]$$

$$= \begin{cases} \sigma_c^{SSY}\left[1-\frac{\pi^2}{48}\left(\frac{\sigma_c^{SSY}}{\sigma_0}\right)^2+\cdots\right], & \text{for Large } a \\ \sigma_0, & \text{for small } a \end{cases}$$



The J integral

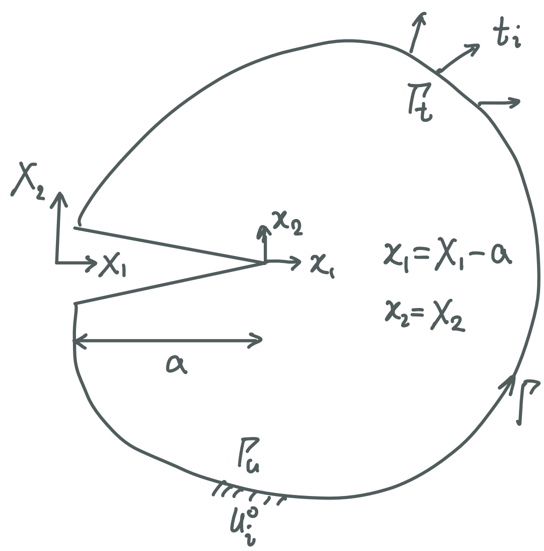
We will show that J is a path-independent integral and is equal to the energy release rate in nonlinear elastic materials.

First, define the strain energy density in a nonlinear material as

$$W = \int_0^{\varepsilon} \sigma_{ij}\, d\varepsilon_{ij} \rightarrow \sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}, \text{ i.e., } W = \int_0^{\varepsilon} \frac{\partial W}{\partial \varepsilon_{ij}}\, d\varepsilon_{ij} = \int_0^{\varepsilon} dW$$

So our governing equations for a small deformation problem in this nonlinear material are

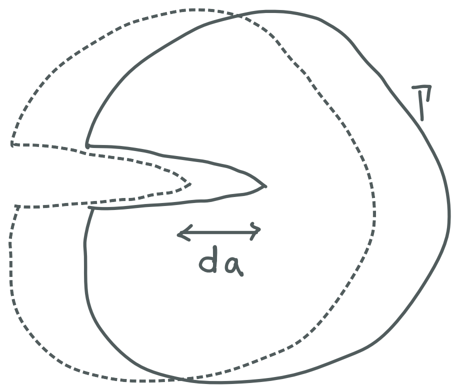
$$\begin{cases} \sigma_{ij,j} = 0 & \text{(Equilibrium)} \\ \sigma_{ji} n_j = t_i \text{ on } \Gamma_t & \\ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) & \text{(Kinematics)} \\ u_i = u_i^0 \text{ on } \Gamma_u & \\ \sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} & \text{(Material law)} \end{cases}$$



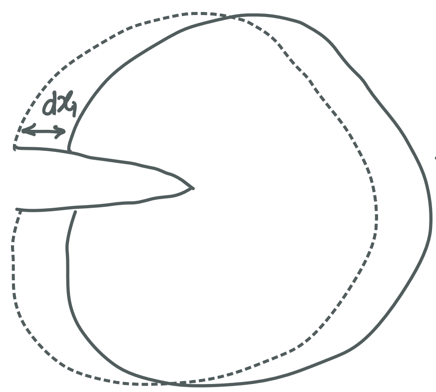
(* We could generalize to large deformation, but we'll not concern ourselves with this complication *)

$$\Pi = \int_A W\, dA - \int_{\Gamma_t} t_i u_i\, d\Gamma$$

$$G = -\frac{D\Pi}{Da}, \quad \text{where } \frac{D}{Da} = \left.\frac{\partial}{\partial a}\right|_{x_1, x_2} + \left.\frac{\partial}{\partial x_1}\right|_a \overbrace{\frac{\partial x_1}{\partial a}}^{-1} = \underbrace{\left.\frac{\partial}{\partial a}\right|_{x_1, x_2}}_{\text{Keep contour stationary \& Move } a \to a+da} - \overbrace{\left.\frac{\partial}{\partial x_1}\right|_a}^{\text{Keep } a \text{ fixed \& move contour to } x_1 \to x_1 + dx_1}$$



Picture for da (Γ remains the same)



Picture for dx_1 (different Γ but same a)

$$\mathcal{G} = -\frac{D}{Da}\int_A W\,dA + \underbrace{\frac{D}{Da}\int_{\Gamma_t} t_i u_i \, d\Gamma} \rightarrow = \int_{\Gamma_t} t_i \frac{Du_i}{Da}\, d\Gamma \quad \text{since} \quad \begin{array}{l} Dt_i = 0 \text{ on } \Gamma_t \\ Du_i = 0 \text{ on } \Gamma_u \\ t_i \equiv 0 \text{ on new crack surfaces due to } Da. \end{array}$$

$$= \frac{\partial}{\partial x_1}\int_A W\,dA - \frac{\partial}{\partial a}\int_A W\,dA + \int_\Gamma t_i\left(\frac{\partial u_i}{\partial a} - \frac{\partial u_i}{\partial x_i}\right) d\Gamma$$

Note that $\frac{\partial}{\partial x_1}$ <u>cannot</u> be taken inside the first integral because the area $A(x_1, x_2)$ has has changed to $A(x_1 + dx_1, x_2)$ as the contour moves. However, $\frac{\partial}{\partial a}$ <u>can</u> be taken inside the second integral, i.e., $\frac{\partial}{\partial a}\int_A W\,dA = \int_A \frac{\partial W}{\partial a}\,dA$.

For crack tip solutions, even in nonlinear elastic materials, we want finite energy in any finite area around the crack tip.

$$W\,dA = W\,r\,dr\,d\theta \text{ finite} \rightarrow W \sim \frac{1}{r}, \quad r = (x_1^2 + x_2^2)^{1/2}$$

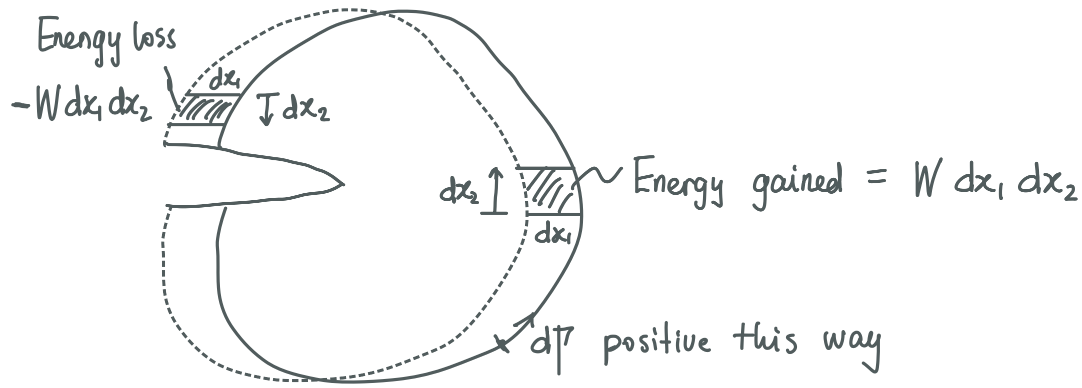
This also means that $\frac{\partial W}{\partial a} \sim \frac{1}{r}$ and $\frac{\partial W}{\partial a}\,dA$ is NOT singular. So, we can apply divergence theorem to this term.

$$\int_A \frac{\partial W}{\partial a}\,dA = \int_A \frac{\partial W}{\partial \varepsilon_{ij}}\frac{\partial \varepsilon_{ij}}{\partial a}\,dA = \int_A \sigma_{ij}\frac{\partial u_{i,j}}{\partial a}\,dA = \int_A \left[\left(\sigma_{ij}\frac{\partial u_i}{\partial a}\right)_{,j} - \cancelto{0}{\sigma_{ij,j}}\frac{\partial u_i}{\partial a}\right] dA = \int_\Gamma \sigma_{ij}\frac{\partial u_i}{\partial a} n_j\, d\Gamma$$

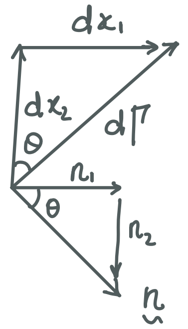
Cancel out

$$\rightarrow \mathcal{G} = \frac{\partial}{\partial x_1}\int_A W\,dA \;\cancel{- \int_\Gamma \sigma_{ij} n_j \frac{\partial u_i}{\partial a}\,d\Gamma + \int_\Gamma t_i \frac{\partial u_i}{\partial a}\,d\Gamma} - \int_\Gamma t_i \frac{\partial u_i}{\partial x_1}\,d\Gamma$$

Now, consider $\frac{\partial}{\partial x_1}\int_A W\,dA$



dx_1 is uniform over the entire contour. $\rightarrow \Delta W = \int W\,dx_1 dx_2 = dx_1 \underbrace{\int W\,dx_2}$



$$dx_2 = d\Gamma\cos\theta = n_1\,d\Gamma$$

Automatically, n_1 take cares of the sign for "energy loss" versus "energy gained" by following the contour.

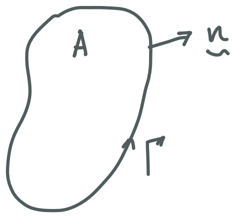
$$\rightarrow \Delta W = dx_1 \int_\Gamma W n_1\,d\Gamma \;\rightarrow\; \frac{\partial}{\partial x_1}\int_A W\,dA = \lim_{dx_1\to 0} \frac{\int_{A(x_1+dx_1)} W\,dA - \int_{A(x_1)} W\,dA}{dx_1}$$

$$= \lim_{dx_1\to 0}\frac{\Delta W}{dx_1} = \int_\Gamma W n_1\,d\Gamma$$

Finally, we have

$$\mathcal{G} = \int_\Gamma W n_1 - \underbrace{t_i \frac{\partial u_i}{\partial x_1}}_{\sigma_{ij} n_j u_{i,1}}\,d\Gamma = J \qquad \text{(J.R. Rice JAM, 1968)}$$

Now show J is path-independent. Consider J around any contour not surrounding a singularity.

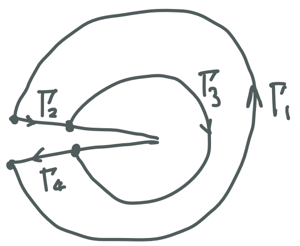


$$J = \int_\Gamma W n_1 - \sigma_{ij} n_j u_{i,1} \, d\Gamma$$

$$\int_\Gamma W n_1 \, d\Gamma = \int_A W_{,1} \, dA = \int_A \frac{\partial W}{\partial \varepsilon_{ij}} u_{i,j1} \, dA = \int_A (\sigma_{ij} u_{i,1})_{,j} \, dA$$

$$= \int_\Gamma \sigma_{ij} u_{i,1} n_j \, dA \rightarrow J = 0$$

Back to our cracked body



$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 \quad \rightarrow J_\Gamma = 0$

A closed contour enclosing no sigularity

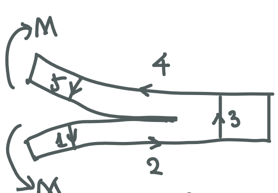
Note that on Γ_2, Γ_4, $n_1 = 0$, $t_i = 0 \rightarrow J_{\Gamma_2} = J_{\Gamma_4} = 0$ & $J_{\Gamma_1} + J_{\Gamma_3} = 0$

$$\rightarrow \mathcal{G} = J_{\Gamma_1} = -J_{\Gamma_3}$$

Any arbitrary contour surrounding the crack tip counterclockwisely (for crack growing to the right)

$$\rightarrow \mathcal{G} = J \text{ \& } J \text{ is path-independent}$$

Examples from HW1: $\quad J = \int_\Gamma W n_1 - \sigma_{ij} n_j u_{i,1} \, d\Gamma$



$B = \frac{1}{12} E h^3 = E I_z / b$

Γ_3: $n_1 = 1$, $W = 0$, $\sigma_{ij} = 0$

$\Gamma_{2,4}$: $n_1 = 0$, $t_i = 0$

from the neutural axis of the arm (upward positive)

Γ_1 : $n_1 = -1$, $\varepsilon_{xx} = \kappa y = \frac{M y}{E I_z}$, $\sigma_{xx} = E \kappa y$, all other $\varepsilon_{ij}, \sigma_{ij} = 0$

$$W = \frac{1}{2}\varepsilon_{xx}\sigma_{xx} = \frac{1}{2}EK^2y^2 \qquad d\Gamma = d(-y)$$

$$J_1 = \int_{\frac{h}{2}}^{-\frac{h}{2}} \frac{1}{2}EK^2y^2\cdot(-1) - EKy\cdot(-1)\cdot Ky \; d(-y)$$

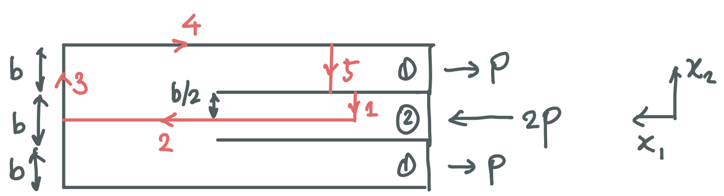
$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}EK^2y^2\,dy = \frac{1}{2}\frac{EI_z}{b}K^2 = \frac{1}{2}BK^2 = \frac{M^2}{2EI_zb}$$

Γ_5: $n_1 = -1$, $\varepsilon_{xx} = Ky$ (taking y from the neutral axis of the arm)

$$\rightarrow J_5 = \frac{1}{2}BK^2 = \frac{M^2}{2EI_zb}$$

$$J = J_1 + J_2 + J_3 + J_4 + J_5 = BK^2 = \frac{M^2}{EI_zb}$$

$\mathcal{G} = \mathcal{G}_c \rightarrow K = \left(\frac{\mathcal{G}_c}{B}\right)^{1/2}$ A jump in curvature since Γ_1, Γ_5 could be taken to close to the crack tip.



The x_1 direction must be choosen in the direction of crack advance. Or you can use "clockwise" contour.

Γ_1: $n_1 = -1$, $n_2 = 0$, $\sigma_{11} = -\frac{2P}{bt}$, $\varepsilon_{11} = u_{1,1} = -\frac{2P}{E_1'bt}$, $d\Gamma = -dx_2$

$$J_1 = \int_{b/2}^{0}\left[\frac{4P^2}{2E_1'b^2t^2}(-1) - \frac{2P}{bt}\left(-\frac{2P}{E_1'bt}\right)\right]d(-x_2) = \frac{2P^2}{E_1'b^2t^2}\cdot\frac{b}{2} = \frac{P^2}{E_1'bt^2}$$

Γ_2: $n_1 = 0$, $n_2 = -1$, $\sigma_{12} = 0$ (symmetry), $d\Gamma = dx_1$

$$J_2 = \int_0^{\ell} 0 - \sigma_{22}\cdot(-1)\,\cancel{u_{2,1}}^{\,0}\,dx_1 = 0$$

Γ_3: $n_1 = 1$, $n_2 = 0$, $\sigma_{ij} = 0 \rightarrow J_3 = 0$

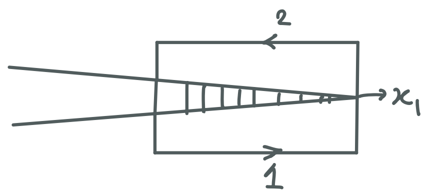
$\Gamma_4: \quad n_1 = 0, \quad t_i = 0 \quad \rightarrow J_4 = 0$

$\Gamma_5: \quad n_1 = -1, \quad n_2 = 0, \quad \sigma_{11} = -\frac{P}{bt}, \quad \varepsilon_{11} = u_{1,1} = -\frac{P}{E_2' bt}, \quad d\Gamma = -dx_2$

$$J_5 = \int_b^0 \left[\frac{P^2}{2E_2' b^2 t^2} \cdot (-1) - \frac{P}{bt}\left(-\frac{P}{E_2' bt}\right) \right] d(-x_2) = \frac{P^2}{2E_2' b t^2}$$

$$\rightarrow \boxed{J = J_1 + J_5 = \frac{P^2}{bt^2}\left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{2E_2}\right)}$$

<u>Consider the cohesive zone model:</u>



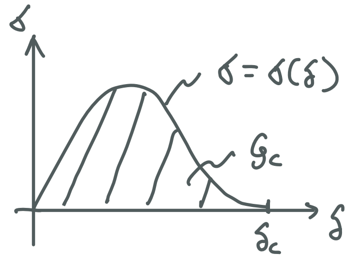
Take Γ to run exactly along the crack faces where the cohesive tractions act

$$J = \int_{-R}^{0} -\overset{\sigma_0}{\sigma_{22}}(-1)\, u^-_{2,1}\, dx_1 + \int_0^{-R} -\overset{\sigma_0}{\sigma_{22}}\, u^+_{2,1}\, d(-x_1)$$

$$= \sigma_0 \int_0^{-R} \overbrace{(u_2^+ - u_2^-)}^{\delta}{}_{,1}\, dx_1$$

$$= \sigma_0 \int_0^{-R} \frac{d\delta}{dx_1}\, dx_1 = \sigma_0\, \delta \Big|_0^{-R} = \sigma_0 \delta_c$$

For a more general cohesive zone model, we have



$$J = \int_{-R}^{0} \sigma_{22}\, u^-_{2,1}\, dx_1 + \int_0^{-R} \sigma_{22}\, u^+_{2,1}\, dx_1$$

$$= \int_0^{-R} \sigma_{22}\, (u_2^+ - u_2^-)_{,1}\, dx_1$$

$$= \int_0^{-R} \sigma(\delta) \frac{d\delta}{dx_1}\, dx_1 = \int_{\delta(x_1=0)}^{\delta(x_1=-R)} \sigma(\delta)\, d\delta = \int_0^{\delta_c} \sigma(\delta)\, d\delta$$

- Under SSY conditions and Mode I, $\underline{\underline{J \equiv \mathcal{G}_c}} = \frac{K_I^2}{E'}$. It does not depend on what cohesive zone law looks like.

- Under LSY conditions, we can still take a J contour to closely surround the fracture process zone and obtain

$$J = \int_0^{\delta_t} \sigma(\delta)\, d\delta$$

where δ_t is the opening at the back edge of the process zone (this expression for J is valid prior to propagation). Propagation occurs when $\underline{\underline{\delta_t = \delta_c}}$, corresponding to

$$J = \int_0^{\delta_c} \sigma(\delta)\, d\delta = \mathcal{G}_c$$

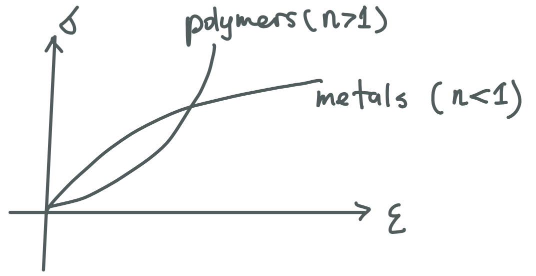
Hence, this model predicts propagation when J reaches $\mathcal{G}_c$. This is equivalent to a fracture criterion based on a critical crack opening displacement. Also note that under LSY conditions, $J \neq \frac{K_I^2}{E'}$. For Dugdale model in the center-cracked panel:

$$J = \frac{K_I^2}{E'^2}\left[1 + \frac{\pi^2}{24}\left(\frac{\sigma}{\sigma_0}\right)^2 + \cdots\right] > J_{SSY}$$

Therefore, J is powerful tool when dealing with nonlinearities and LSY.

The HRR fields

Hutchinson JMPS v.16 pp. 13–31 (1968); Rice & Rosengren JMPS v.16. pp 1–12 (1968)



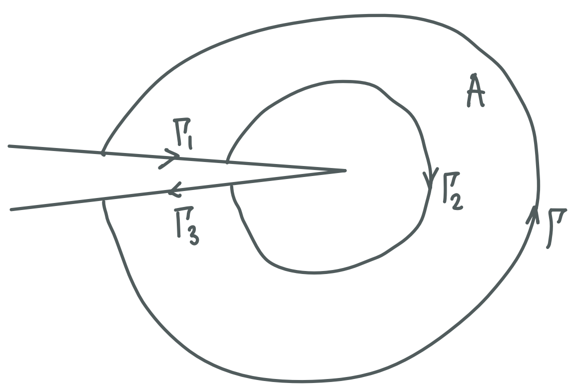
$$\frac{\varepsilon}{\varepsilon_Y} = \frac{\sigma}{\sigma_Y} + \alpha\left(\frac{\sigma}{\sigma_Y}\right)^n$$

$n \to 1 \to$ linear law

$$\begin{cases} \dfrac{\sigma_{ij}}{\sigma_Y} = \left(\dfrac{J}{\alpha\,\varepsilon_Y \sigma_Y I_n}\,\dfrac{1}{r}\right)^{\frac{1}{n+1}} \hat{\sigma}_{ij}(\theta) \\[2ex] u_i = \alpha\,\varepsilon_Y\, r \left(\dfrac{J}{\alpha\,\varepsilon_Y \sigma_Y I_n}\,\dfrac{1}{r}\right)^{\frac{n}{n+1}} \hat{u}_i(\theta) \end{cases}$$

Universal angular functions

<u>Domain integral method for J calculations</u> (useful in FEM)



$$J = \int_\Gamma \left(W n_1 - \sigma_{ji}\, n_j\, u_{i,1}\right) d\Gamma$$

Define a sufficiently smooth $q(x_1, x_2)$ such that $q = 1$ on Γ and $q = 0$ on Γ_2.

Since $J_{\Gamma_1} = J_{\Gamma_3} = 0$ ($n_1 = 0$, $t_i = 0$ on crack faces), we have

$$J = \int_{\Gamma + \Gamma_1 + \Gamma_2 + \Gamma_3} \left(W q n_1 - q\,\sigma_{ji}\, n_j\, u_{i,1}\right) d\Gamma$$

$$= \int_A \left(\cancel{\frac{\partial W}{\partial x_1} q} + W \frac{\partial q}{\partial x_1} - q_{,j}\,\sigma_{ji}\, u_{i,1} - q\, \overset{0}{\cancel{\sigma_{ji,j}}}\, u_{i,1} - \cancel{q\,\sigma_{ji}\, u_{i,j1}}\right) d\Gamma$$

$$\rightarrow \quad \boxed{J = \int_A \left(W q_{,1} - \sigma_{ji}\, q_{,j}\, u_{i,1}\right) dA}$$