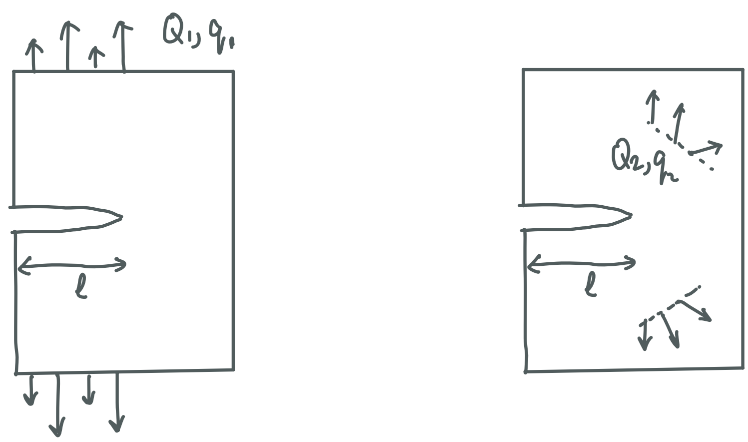
Weight Function Methods

(Ref: Rice, IJSS, 1973, P751)

Consider a linear elastic body with a crack on the x-axis and symmetric loading and elastic properties about the x-axis (it can be anisotropic but orthotropic with material directions aligned with the x & y axes). We are interested in mode I but the method can be generalized to other modes and more general anisotropy.

Now, consider two different load systems.



Q_1, Q_2 are generalized forces such that the tractions and body forces are

$$t_i = Q_1 t_i^{(1)}, \quad b_i = Q_1 b_i^{(1)} \quad \text{in problem 1}$$

and

$$t_i = Q_2 t_i^{(2)}, \quad b_i = Q_2 b_i^{(2)} \quad \text{in problem 2.}$$

So, you may think of Q_i's as scaling factors. Then q_i's are generalized displacements such that Q_i and q_i form a work-conjugate pair for

u_i^* (i.e., any deformation field).

$$Q_1 q_1 = \int_S t_i u_i^* ds + \int_V b_i u_i^* dV = Q_1 \int_S t_i^{(1)} u_i^* ds + Q_1 \int_V b_i^{(1)} u_i^* dV$$

$$\rightarrow q_1 = \int_S t_i^{(1)} u_i^* ds + \int_V b_i^{(1)} u_i^* dV$$

and similarly $q_2 = \int_S t_i^{(2)} u_i^* ds + \int_V b_i^{(2)} u_i^* dV$

If both load systems act simultaneously, then we will write the total displacement as

$$u_i = Q_1 u_i^{(1)} + Q_2 u_i^{(2)}$$

where $u_i^{(1)}$ is the displacement field due to a unit Q_1, and $u_i^{(2)}$ is the displacement field due to a unit Q_2. At this moment, we have

$$q_1 = \int_S t_i^{(1)} \left[Q_1 u_i^{(1)} + Q_2 u_i^{(2)} \right] ds + \int_V b_i^{(1)} \left[Q_1 u_i^{(1)} + Q_2 u_i^{(2)} \right] dV$$

$$= \underbrace{\left[\int_S t_i^{(1)} u_i^{(1)} ds + \int_V b_i^{(1)} u_i^{(1)} dV \right]}_{C_{11}} Q_1 + \underbrace{\left[\int_S t_i^{(1)} u_i^{(2)} ds + \int_V b_i^{(1)} u_i^{(2)} dV \right]}_{C_{12}} Q_2$$

Similarly

$$q_2 = \underbrace{\left[\int_S t_i^{(2)} u_i^{(2)} ds + \int_V b_i^{(2)} u_i^{(2)} dV \right]}_{C_{22}} Q_2 + \underbrace{\left[\int_S t_i^{(2)} u_i^{(1)} ds + \int_V b_i^{(2)} u_i^{(1)} dV \right]}_{C_{21}} Q_1$$

$\rightarrow q_i = C_{ij} Q_j$ or $Q_i = C_{ij}^{-1} q_j$ ($C_{ij} = C_{ji}$ due to Rayleigh-Betti reciprocal theorem).

Note that $C_{ij} = C_{ij}(l)$ are structural compliances, depending on the crack length l.

Now the stored strain energy in the body at fixed generalized displacement is

$$U(q_1, q_2, l) = \frac{1}{2} Q_i q_i = \frac{1}{2} C_{ij}^{-1}(l)\, q_i q_j$$

where $\dfrac{\partial U}{\partial q_1} = C_{11}^{-1} q_1 + C_{12}^{-1} q_2 = Q_1$

$$\frac{\partial U}{\partial q_2} = C_{21}^{-1} q_1 + C_{22}^{-1} q_2 = Q_2$$

$$\frac{\partial U}{\partial l} = \frac{1}{2} \frac{\partial C_{ij}^{-1}(l)}{\partial l} q_i q_j = -\mathcal{G} t \quad (\text{fixed } q_i)$$

↑ thickness in the out-of-plane direction

The potential energy of the system is a function of the Q_i and l and is the strain energy minus the work done by the loads

$$P.E. = \psi = U - Q_i q_i = U - Q_1 q_1 - Q_2 q_2$$

$$d\psi = \underbrace{\frac{\partial U}{\partial q_1}}_{Q_1} dq_1 + \underbrace{\frac{\partial U}{\partial q_2}}_{Q_2} dq_2 + \underbrace{\frac{\partial U}{\partial l}}_{-\mathcal{G}t} dl - dQ_1 q_1 - Q_1 dq_1 - dQ_2 q_2 - Q_2 dq_2$$

$$= \underbrace{-\mathcal{G}t}_{\frac{\partial \psi}{\partial l}}\, dl \underbrace{-\, q_1}_{\frac{\partial \psi}{\partial Q_1}} dQ_1 \underbrace{-\, q_2}_{\frac{\partial \psi}{\partial Q_2}} dQ_2$$

$$\boxed{\begin{array}{l} \left.\dfrac{\partial \psi}{\partial l}\right|_{Q_1, Q_2} = -\mathcal{G}t \ \checkmark \\ -\dfrac{\partial \psi}{\partial Q_1} = q_1 , \quad -\dfrac{\partial \psi}{\partial Q_2} = q_2 \end{array}}$$

Now define k_1 to be the stress intensity factor for problem 1 when $Q_1 = 1$.

k_2 to be the SIF for problem 2 when $Q_2 = 1$. Then, due to linear superposition,

$$K = k_1 Q_1 + k_2 Q_2 , \quad \mathcal{G} = \frac{1}{H}\left(k_1 Q_1 + k_2 Q_2\right)^2 , \quad (H \text{ is an elastic modulus}$$

$H = E'$ for isotropic materials).

Our goal in this business is to determine $\underline{\underline{k_2}}$ given that we have a complete solution for problem $\underline{\underline{1}}$!

- $$\frac{\partial(\mathcal{G}t)}{\partial Q_i} = -\frac{\partial^2 \Psi}{\partial Q_i \partial l} = -\frac{\partial}{\partial l}\left(\frac{\partial \Psi}{\partial Q_i}\right) = +\frac{\partial q_i}{\partial l} = \frac{\partial C_{ij}}{\partial l} Q_j$$

$q_i = C_{ij}(l)\, Q_j$

a scale factor independent of l.

- $$\frac{\partial(\mathcal{G}t)}{\partial Q_i} = \frac{\partial}{\partial Q_i}\left[t\,\frac{(k_j Q_j)^2}{H}\right] = \frac{2t}{H} k_j Q_j \cdot \left(k_j \delta_{ij}\right) = \frac{2t}{H} k_i k_j Q_j$$

$$\rightarrow \left(\frac{2t}{H} k_i k_j - \frac{\partial C_{ij}}{\partial l}\right) Q_j = 0$$

Here, we can take $Q_1 = 1$, $Q_2 = 0$ or $Q_1 = 0$, $Q_2 = 1$, or any other combinations and this relationship holds.

$$\rightarrow \frac{\partial C_{ij}}{\partial l} = \frac{2t}{H} k_i k_j$$

Consider the cross term, $\frac{2t}{H} k_1 k_2 = \frac{\partial C_{12}}{\partial l} = \frac{\partial C_{21}}{\partial l}$

$$\rightarrow k_2 = \frac{H}{2t} \underbrace{\frac{1}{k_1 Q_1}}_{K \text{ for problem 1}} \frac{\partial}{\partial l}\left(C_{21} Q_1\right)$$

Therefore, K due to problem 2 is

$$K^{(2)} = k_2 Q_2 = \frac{H}{2t} \frac{1}{K^{(1)}} \cdot \frac{\partial}{\partial l}\left(C_{21} Q_1\right) \cdot Q_2$$

$$= \frac{H}{2t} \frac{1}{K^{(1)}} Q_2 \frac{\partial}{\partial l}\left[\int_S t_i^{(2)} Q_1 u_i^{(1)} ds + \int_V b_i^{(2)} Q_1 u_i^{(1)} dV\right]$$

$t_i^{(2)}, b_i^{(2)}$ do not depend on l

$$K^{(2)} = Q_2 \frac{H}{2t} \frac{1}{K^{(1)}} \left[\int_S t_i^{(2)} \frac{\partial Q_1 u_i^{(1)}}{\partial l} ds + \int_V b_i^{(2)} \frac{\partial Q_1 u_i^{(1)}}{\partial l} dV\right]$$

But $K^{(2)}$ should not depend on how the loading in problem 1 is specified.

This means the quantity $\frac{1}{K^{(1)}} \frac{\partial Q_1 u_i^{(1)}}{\partial l}$ should be universal for the given geometry

Define the weight function h_i as

$$h_i = \frac{H}{2t} \frac{1}{K^{(1)}} \frac{\partial Q_1 u_i^{(1)}(x,y,l)}{\partial l} = \underbrace{\frac{H}{2t} \frac{1}{K} \cdot \frac{\partial u_i}{\partial l}}$$

superscript is dropped to denote that h_i can be determined from any problem.

$$\rightarrow K^{(2)} = Q_2 \int_S t_i^{(2)} h_i ds + Q_2 \int_V b_i^{(2)} h_i dV$$

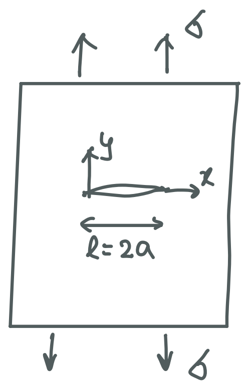
$$\text{or } \boxed{K = \int_S t_i h_i ds + \int_V b_i h_i dV}$$

Note that for 2D problems, it is common to do the surface integral over the the boundary line and the volume integral over the area in which case "t" is dropped. We then have

$$\boxed{K = \int_\Gamma t_i h_i \, d\Gamma + \int_A b_i h_i \, dA \quad , \text{ where } h_i = \frac{H}{2}\frac{1}{K}\frac{\partial u_i}{\partial l} \text{ from another problem}}$$

Usually, the most useful solution to have is that for a pair of point loads opening the crack. The solution can then be used as a Green's function to generate all other solutions using superposition. Let's use weight functions to get such a solution for the center crack.

We know the following solution:



$$K^{(1)} = \sigma\sqrt{\pi a} = \sigma\sqrt{\tfrac{1}{2}\pi l}$$

$$u_y^{(1)}(x, y=\pm 0; \sigma, l) = \pm \frac{2\sigma}{H}\sqrt{x(l-x)}$$

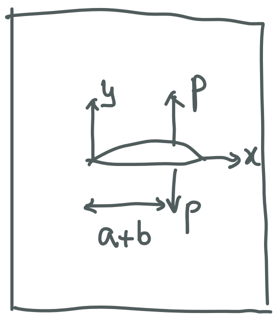
On crack surfaces: $\displaystyle \frac{\partial u_y^{(1)}}{\partial l} = \pm \frac{2\sigma}{H}\frac{\partial}{\partial l}\sqrt{x(l-x)}$

$$= \pm \frac{2\sigma}{H}\frac{1}{2}\frac{x}{\sqrt{x(l-x)}}$$

$$= \pm \frac{\sigma}{H}\sqrt{\frac{x}{l-x}}$$

$$h_y = \frac{H}{2}\frac{1}{K^{(1)}}\frac{\partial u_y^{(1)}}{\partial l} = \pm \frac{\cancel{H}}{2}\frac{1}{\cancel{\sigma}\sqrt{\frac{1}{2}\pi l}} \cdot \frac{\cancel{\sigma}}{\cancel{H}}\sqrt{\frac{x}{l-x}} = \pm \frac{1}{\sqrt{2\pi l}}\sqrt{\frac{x}{l-x}}$$

Now consider a pair of point loads:



$$t_i^{(2)} = \pm P\, \delta(x = a+b,\ y = \pm 0)\, \delta_{i2}$$

$\uparrow$ Dirac delta $\qquad$ $\uparrow$ Kronecker delta

$$b_i^{(2)} = 0$$

$$K^{(2)} = \int_\Gamma t_i^{(2)} h_i \, d\Gamma + \int_A b_i^{(2)} h_i \, dA$$

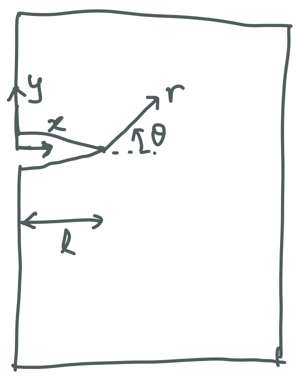
$$= 2 \times P \times \frac{1}{\sqrt{2\pi l}} \cdot \sqrt{\frac{a+b}{\underbrace{l}_{2a} - (a+b)}} = P \frac{1}{\sqrt{\pi a}} \sqrt{\frac{a+b}{a-b}}$$

For which tip? – for the crack tip that appears to grow with increasing l!

$$\rightarrow K^{\text{right}} = \frac{P}{\sqrt{\pi a}} \sqrt{\frac{a+b}{a-b}}$$

$$K^{\text{left}} = \frac{P}{\sqrt{\pi a}} \sqrt{\frac{a-b}{a+b}}$$

Mode I weight functions for a semi-infinite crack



We have derived the asymptotic $u_i(r, \theta)$ for mode I crack tip:

$$u_x = \frac{K_I}{2E'} \sqrt{\frac{r}{2\pi}} (1+\nu) \left[(2\kappa - 1) \cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \right]$$

$$u_y = \frac{K_I}{2E'} \sqrt{\frac{r}{2\pi}} (1+\nu) \left[(2\kappa + 1) \sin\frac{\theta}{2} - \sin\frac{3\theta}{2} \right]$$

How to perform $\frac{\partial u_i}{\partial l}$? Geometry: $r = \sqrt{(x-l)^2 + y^2}$, $\theta = \arctan \frac{y}{x-l}$

$$\frac{\partial r}{\partial l} = \frac{(x-l)\times(-1)}{\sqrt{(x-l)^2+y^2}} = -\cos\theta$$

$$\frac{\partial \theta}{\partial l} = \frac{1}{1+\left(\frac{y}{x-l}\right)^2} \frac{y}{(x-l)^2} = \frac{y}{r^2} = \frac{\sin\theta}{r}$$

$$h_x^{I} = \frac{E'}{2}\frac{1}{K_I}\frac{\partial u_x}{\partial l}$$

$$= \frac{E'}{2}\frac{1}{K_I}\cdot\frac{K_I}{2E'}\frac{(1+\nu)}{\sqrt{2\pi}}\left\{\frac{1}{2\sqrt{r}}\cdot(-\cos\theta)\left[(2K-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right]\right.$$

$$\left. + \sqrt{2}\left[(2K-1)\left(-\sin\frac{\theta}{2}\cdot\frac{1}{2}\right) + \sin\frac{3}{2}\theta\cdot\frac{3}{2}\right]\cdot\frac{\sin\theta}{r}\right\}$$

$$= \frac{1}{K+1}\frac{1}{\sqrt{2\pi r}}\left[(1-K)\cos\frac{\theta}{2} + \sin\theta\,\sin\frac{3\theta}{2}\right]$$

Similarly, we have $h_y^{I} = \frac{1}{K+1}\frac{1}{\sqrt{2\pi r}}\left[(K+1)\sin\frac{\theta}{2} - \sin\theta\,\cos\frac{3\theta}{2}\right]$

In HW3, you will show Mode II weight functions for a semi-infinite crack.

$$h_x^{II} = \frac{1}{K+1}\frac{1}{\sqrt{2\pi r}}\left[(K+1)\sin\frac{\theta}{2} + \cos\frac{3}{2}\theta\,\sin\theta\right]$$

$$h_y^{II} = \frac{1}{K+1}\frac{1}{\sqrt{2\pi r}}\left[(K-1)\cos\frac{\theta}{2} + \sin\frac{3}{2}\theta\,\sin\theta\right]$$