Determination of K

Having known the K–G relation, the next question to answer is how to determine K for a given elasticity problem.

Answer: Use stresses in front of the crack tip or the crack opening or shearing/sliding displacements behind the crack tip.

$$K_I = \lim_{r \to 0} \sqrt{2\pi r}\, \sigma_{yy}(r, \theta=0)$$

$$K_{II} = \lim_{r \to 0} \sqrt{2\pi r}\, \sigma_{xy}(r, \theta=0)$$

$$K_{III} = \lim_{r \to 0} \sqrt{2\pi r}\, \sigma_{yz}(r, \theta=0)$$

True for both isotropic & anisotropic materials

or

True only for isotropic materials

$$K_I = \lim_{r \to 0} \frac{E'}{8} \sqrt{\frac{2\pi}{r}} \left[u_y(r, \theta=\pi) - u_y(r, \theta=-\pi) \right]$$

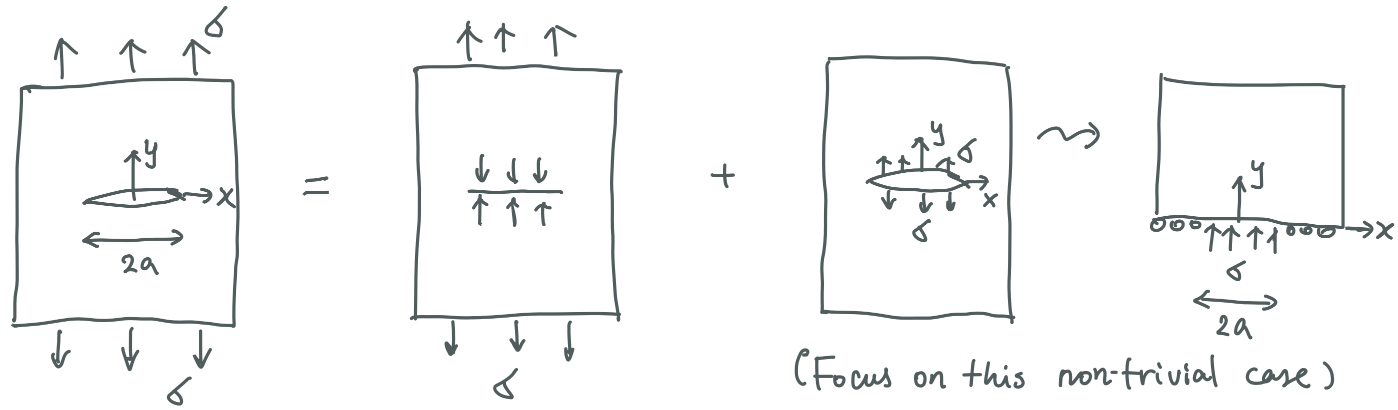
$$K_{II} = \lim_{r \to 0} \frac{E'}{8} \sqrt{\frac{2\pi}{r}} \left[u_x(r, \theta=\pi) - u_x(r, \theta=-\pi) \right]$$

$$K_{III} = \lim_{r \to 0} \frac{\mu}{4} \sqrt{\frac{2\pi}{r}} \left[u_z(r, \theta=\pi) - u_z(r, \theta=-\pi) \right]$$

Then the rest would be to solve for $\underset{\sim}{u}$ or $\underset{\sim}{\sigma}$ from (mixed) boundary value problem with given geometry and boundary conditions. Accordingly, we can apply methods such as separation of varibles (HW2), transform method, Wiener–Hof technique (To deal with non-vanishing ends), Green's function (used later), conformal mapping (limited success), complex varible method (will be used extensively), and DNS.

Transform method

Let's consider the center crack problem again but solve it by transform method.



(Focus on this non-trivial case)

Governing equation: $\nabla^2\nabla^2\phi = 0 \quad \left(= \nabla^4 \overset{u_1}{u} = \nabla^4 \overset{u_2}{v}\right)$

Boundary conditions: $\tau_{xy} = 0$ on $y=0$, $\sigma_{yy} = -\sigma$ on $y=0$, $|x|<a$

$v = 0$ on $y=0$, $|x|>a$

all terms $\to 0$ as $y \to \infty$

Fourier transform: $\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx = H(f)$

Inverse FT: $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\, e^{-ikx}\, dk = H^{-1}(f)$

Property: $\int_{-\infty}^{\infty} \frac{df}{dx} e^{ikx} dx = f e^{ikx}\Big|_{-\infty}^{\infty} - ik\int_{-\infty}^{\infty} f e^{ikx} dx \to \tilde{f'} = -ik\tilde{f}$

$$\to H\left(\nabla^2(\nabla^2\phi)\right) = \left(\frac{d^2}{dy^2} - k^2\right)^2 \tilde{\phi} = 0$$

The solution can be written in the form

$$\tilde{\phi}(k,y) = (A+By)e^{-|k|y} + (C+Dy)e^{+|k|y}$$

To determine integration constants, we need to obtain expression of stresses and displacements in term of $\tilde{\phi}$.

$$\tilde{\sigma}_{xx} = \int_{-\infty}^{\infty} \sigma_{xx} e^{ikx} dx = \int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial y^2} e^{ikx} dx = \frac{\partial^2 \tilde{\phi}}{\partial y^2}$$

$$\tilde{\sigma}_{yy} = \int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial x^2} e^{ikx} dx = -k^2 \tilde{\phi}$$

$$\tilde{\sigma}_{xy} = \int_{-\infty}^{\infty} -\frac{\partial^2 \phi}{\partial x \partial y} e^{ikx} dx = +ik \frac{\partial \tilde{\phi}}{\partial y}$$

As $y \to \infty$, $\sigma_{ij} \to 0 \rightarrow C = D = 0$

even function of k

At $y=0$, $\sigma_{yy} = -p(x) \Rightarrow -k^2 \tilde{\phi}\big|_{y=0} = -Ak^2 = -\tilde{p}(k) \rightarrow A = \tilde{p}(k)/k^2$

Arbitrary even function

$$\sigma_{xy} = 0 \Rightarrow \frac{\partial \tilde{\phi}}{\partial y}\bigg|_{y=0} = -|k| A + B = 0 \rightarrow B = \tilde{p}(k)/|k|$$

$$\rightarrow \tilde{\phi} = \frac{\tilde{p}(k)}{k^2} (1 + |k| y) e^{-|k| y}$$

$$\tilde{\sigma}_{xx} = -\tilde{p}\,(1 - |k| y) e^{-|k| y}$$

$$\tilde{\sigma}_{yy} = -\tilde{p}\,(1 + |k| y) e^{-|k| y}$$

$$\tilde{\sigma}_{xy} = -i\tilde{p}\, k y\, e^{-|k| y}$$

Odd of k

Once we know $p(x)$, we can calculate $\tilde{p}(k)$ and then $\tilde{\sigma}_{ij}$ & σ_{ij}. While $p(x) = \sigma$ for $|x| < a$, $p(x)$ for $|x| > a$ is also part of the solution to ensure $v(x) = 0$. We then need expressions for $\tilde{u}$ and $\tilde{v}$ as well.

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E'}\left(\sigma_{xx} - \nu'\sigma_{yy}\right) \rightarrow -ik\tilde{u} = \frac{1}{E'}\left(\tilde{\sigma}_{xx} - \nu'\tilde{\sigma}_{yy}\right)$$

$$\rightarrow \tilde{u} = \frac{i\tilde{p}}{E'k}\left[-(1-\nu') + (1+\nu')|k|y\right]e^{-|k|y}$$

↖ Odd function of k

[Typo in Sneddon's book]

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1+\nu'}{E'}\sigma_{xy} \rightarrow \frac{\partial \tilde{u}}{\partial y} - ik\tilde{v} = \frac{2(1+\nu')}{E'}\tilde{\sigma}_{xy}$$

$$\rightarrow \tilde{v} = \frac{\tilde{p}}{E'|k|}\left[2 + (1+\nu')|k|y\right]e^{-|k|y}$$

↖ Even function of k

We are particularly interested in σ_{yy} and v, associated with which BCs are.

$$\sigma_{yy} = \frac{-1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\tilde{p}(1+|k|y)e^{-|k|y}}_{\text{Even of } k} e^{-ikx}\,dk = -\frac{1}{\pi}\int_0^\infty \tilde{p}(1+ky)e^{-ky}\cos kx\,dk$$

$$v = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\tilde{p}}{E'|k|}\left[2+(1+\nu')|k|y\right]e^{-|k|y}e^{-ikx}\,dk = \frac{1}{\pi E'}\int_0^\infty \tilde{p}\left[2+(1+\nu')ky\right]e^{-ky}\frac{\cos kx}{k}\,dk$$

On the surface of $y=0$, we have

$$\sigma_{yy} = -\frac{1}{\pi}\int_0^\infty \tilde{p}(k)\cos kx\,dk, \quad v(x) = \frac{2}{\pi E'}\int_0^\infty \tilde{p}(k)\frac{\cos kx}{k}\,dk$$

We see therefore that the equations deteming $\tilde{p}(k)$ are the dual integral equations

$$\frac{2}{\pi}\int_0^\infty \tilde{p}(k)\cos kx\,dk = p(x), \quad 0 \le x \le a$$

$$\int_0^\infty \tilde{p}(k)\frac{\cos kx}{k}\,dk = 0, \quad x > a$$

The duel integral equations (often in mixed BVP) need to solved numerically in general.

However, for this problem, we can make use of Busbridge's solution.

$$K = k\cdot a\,,\quad X = x/a\,,\quad \tilde{P} = \tilde{p}/K^{1/2}\,,\quad P = a\left(\frac{\pi}{2X}\right)^{1/2} p_0\,,\quad \cos KX = \left(\frac{\pi K X}{2}\right)^{1/2} J_{-1/2}(KX)$$

$$\rightarrow \begin{cases} \int_0^\infty K\tilde{P}(K)\, J_{-1/2}(KX)\, dK = P(X)\,, & 0 \le X \le 1 \\ \int_0^\infty \tilde{P}(K)\, J_{-1/2}(KX)\, dK = 0\,, & X > 1 \end{cases}$$

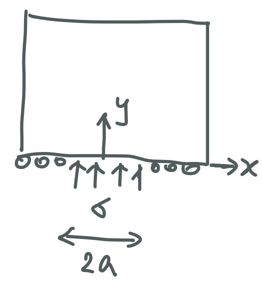
Sneddon showed the solution (Page 424)

$$\tilde{P}(K) = \left(\frac{2K}{\pi}\right)^{1/2}\left[J_0(K)\int_0^1 X^{1/2}(1-X^2)^{-1/2} P(X)\,dX + K\int_0^1 X^{1/2}(1-X^2)^{1/2}dX\int_0^1 P(\alpha X)\,\alpha^{\frac{5}{2}} J_1(K\alpha)\,d\alpha\right]$$

When $p(x) = \sigma$ within $|x| < 1$, $\tilde{p}(k) = \frac{1}{2}\pi\sigma a J_1(ka)$

$$\boxed{\begin{aligned} v(y=0) &= \frac{2}{E'}\sigma\left(a^2 - x^2\right)^{1/2} \\ \sigma_{yy}(y=0, x>c) &= \sigma\left[\frac{x}{(x^2-a^2)^{1/2}} - 1\right] \end{aligned}}$$

for



for the problem by adding the trivial solution.

We then can calculate the stress intensity factor by

$$K_I = \lim_{r\to 0}\sqrt{2\pi r}\,\sigma_{yy}(r, \theta=0) \overset{x=r+a}{=\!=} \lim_{r\to 0}\sqrt{2\pi r}\,\sigma\cdot\frac{r+a}{\sqrt{r+2a}\sqrt{r}} = \sqrt{\pi a}\,\sigma.$$

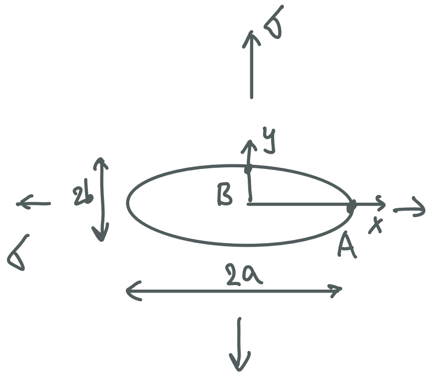
or

$$K_I = \lim_{r\to 0}\frac{E'}{8}\sqrt{\frac{2\pi}{r}}\, u_y(r, \theta=\pi)\times 2 \overset{x=a-r}{=\!=} \lim_{r\to 0}\frac{E'}{4}\sqrt{\frac{2\pi}{r}}\cdot\frac{2}{E'}\sigma\sqrt{2a+r}\sqrt{r} = \sqrt{\pi a}\,\sigma$$

symmetry

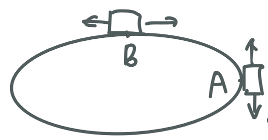
$r = a - x$

For this problem, the crack opening displacement can also be calculated by considering the following problem:



(Taking $b \to 0$ gives rise to a center crack).

Eshelby: The strain in an ellipsoidal inclusion in an infinite matrix loaded at infinity is uniform.



- From elasticity we know the stress concentration at 2 key points
- Thinking of the void as an "inclusion" with shear modulus $\mu \to 0$

- Using strain continuity conditions at the edge of the void, we can determine the "strain" in the void, i.e.,

$$\epsilon_{yy}^{void} = \epsilon_{yy}^{A/void} = \epsilon_{yy}^{A/matrix} = \frac{2\sigma}{E'}\frac{a}{b} \quad \left(\text{since } \sigma_{xx}^{A/void} = \sigma_{xx}^{A/matrix} = 0\right)$$

$$\epsilon_{xx}^{void} = \epsilon_{xx}^{B/void} = \epsilon_{xx}^{B/matrix} = \frac{2\sigma}{E'}\frac{b}{a} \quad \left(\sigma_{yy}^{A/void} = \sigma_{yy}^{A/matrix} = 0\right)$$

$$\epsilon_{xy}^{void} = 0 \quad (\text{symmetry})$$

- Then the displacement field of the void is

$$u_x = \epsilon_x^{void} \cdot x = \frac{2\sigma}{E'}\frac{b}{a}x, \qquad u_y = \epsilon_y^{void} \cdot y = \frac{2\sigma}{E'}\frac{a}{b}y$$

- The vertical displacement "COD" on the surface $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, where

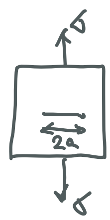
$$y = \pm b\sqrt{1-\left(\frac{x}{a}\right)^2}$$

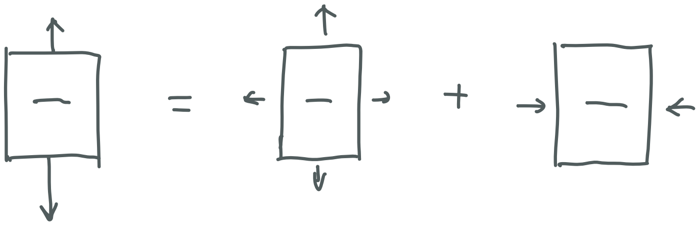
→ on the surface $u_y = \pm \frac{2\sigma}{E'}\sqrt{a^2-x^2}$, $u_x = \frac{2\sigma}{E'}\frac{b}{a}x$

→ Now take $b \to 0$ $u_y = \pm \frac{2\sigma}{E}\sqrt{a^2-x^2}$, $u_x = 0$

$$\boxed{COD(x) = \frac{4\sigma}{E}\sqrt{a^2-x^2}}$$

$$K_I = \lim_{r\to 0} \frac{E'}{8}\sqrt{\frac{2\pi}{r}}\, COD(x) \overset{x=a-r}{=\!=} \lim_{r\to 0} \frac{E'}{4}\sqrt{\frac{2\pi}{r}} \cdot \frac{2}{E'}\sigma\sqrt{2a+r}\sqrt{r} = \sqrt{\pi a}\,\sigma$$

Then what about  K_I ?



uniform σ_{xx} stress field
(Does not contribute to K_I or $G = \frac{\partial W}{\partial a} = 0$)

$$K_I = \sqrt{\pi a}\,\sigma + (K_I = 0) = \sqrt{\pi a}\,\sigma$$

This is nice for a simple problem, but what about more general methods?