Stress field near a crack tip

The previous examples allows us to calculate or measure $\mathcal{G}$ directly. How can we compute $\mathcal{G}$ in general? – Solve the boundary value problem.

For general elasticity problems we must solve the following problem.

Equilibrium: $\sigma_{ji,j} = 0$ in V (no body force)

differentiation wrt the indicies after the comma.

$\sigma_{ij} = \sigma_{ji}$ in V and on S.

$\sigma_{ji} n_j = T_i$ on S

unit normal out of s.

Kinematic: $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

Hooke's law: $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$. or $\epsilon_{ij} = S_{ijkl}\sigma_{kl}$.

First, we'll consider the in-plane loading modes:

Equilibrim: $\dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} = 0$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$

Kinematics: $\epsilon_{11} = \dfrac{\partial u_1}{\partial x_1}$, $\epsilon_{22} = \dfrac{\partial u_2}{\partial x_2}$, $\epsilon_{12} = \epsilon_{21} = \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right)$

Isotropic linear elasticity:

$$\epsilon_{11} = \frac{1}{E'}(\sigma_{11} - \nu'\sigma_{22})$$

$$\epsilon_{22} = \frac{1}{E'}(\sigma_{22} - \nu'\sigma_{11})$$

$$\epsilon_{12} = \frac{1+\nu'}{E'}\sigma_{12}$$

$$E' = \begin{cases} E & \text{plane stress} \\ \frac{E}{1-\nu^2} & \text{plane strain} \end{cases} \qquad \nu' = \begin{cases} \nu & \text{plane stress} \\ \frac{\nu}{1-\nu} & \text{plane strain} \end{cases}$$

To solve these equations we introduce Airy's stress function ϕ such that

$$\sigma_{11} = \frac{\partial^2\phi}{\partial x_2}, \quad \sigma_{12} = -\frac{\partial^2\phi}{\partial x_1 \partial x_2}, \quad \sigma_{22} = \frac{\partial^2\phi}{\partial x_1^2}.$$

Then $\sigma_{11,1} + \sigma_{12,2} = \phi_{,221} - \phi_{,122} = 0$, $\sigma_{12,1} + \sigma_{22,2} = 0$, i.e.. equilibrium equations are automatically satisfied. ϕ is not arbitrary.

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} = u_{1,122} + u_{2,211} - (u_{1,221} + u_{2,112}) = 0$$

Inserting Hooke's law gives

$$\sigma_{11,22} - \nu'\sigma_{22,22} + \sigma_{22,11} - \nu'\sigma_{11,11} - 2(1+\nu')\sigma_{12,12} = 0$$

$$\phi_{,2222} - \cancel{\nu'\phi_{,1122}} + \phi_{,1111} - \cancelto{0}{\nu'\phi_{,2211}} + 2\phi_{,1122} + \cancelto{0}{2\nu'\phi_{,1122}} = 0$$

$$\rightarrow \boxed{\nabla^4\phi = \phi_{,1111} + 2\phi_{,1122} + \phi_{,2222} = 0} \quad \text{Biharmonic equation}$$

We are interested these relations in polar coordinates:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}, \quad \sigma_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}, \quad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$$

$$\nabla^4 \phi = \nabla^2(\nabla^2 \phi) = 0, \quad \text{where } \nabla^2 = \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Known to Michell, solutions to biharmonic equation are

$\ln r$, r^2, $r^2 \ln r$, φ, $\ln r \cdot \varphi$, $r^2\varphi$, $r^2 \ln r \cdot \varphi$, $r\varphi \sin\varphi$, $r \ln r \cos\varphi$, $r \ln r \sin\varphi$

$r^s \cos(s\varphi)$, $r^s \sin(s\varphi)$, $r^{s+2}\cos(s\varphi)$, $r^{s+2}\sin(s\varphi)$ for any real number s

Not all of these solution are useful. For our problem, we want to investigate the field very close to the crack tip. This requires three things:

$\theta \in [-\pi, \pi]$

$r \in (0, \infty)$

- The solution allows discontinuity between $\theta = \pm\pi$ (Contrast to the continuity condition required in precious course)
- At $\theta = \pm\pi$ for all r, $\sigma_{22} = \sigma_{\theta\theta} = 0$ & $\sigma_{12} = \sigma_{r\theta} = 0$ (for not satisfied solutions)
- The energy in a region near the crack tip should be finite to be physical. $U \propto \int_{-\pi}^{\pi}\int_0^r \sigma^2 r\,dr\,d\theta \propto \int_0^r \frac{\phi^2}{r^4} r\,dr \to$ finite.

Therefore, seek solution of the form

$$\phi = \sum_p r^{p+2}\left[A_p \cos p\theta + B_p \cos(p+2)\theta + C_p \sin p\theta + D_p \sin(p+2)\theta\right]$$

$$\begin{cases} \sigma_{\theta\theta} = \sum_p (p+2)(p+1) r^p \left[A_p \cos p\theta + B_p \cos(p+2)\theta + C_p \sin p\theta + D_p \sin(p+2)\theta \right] \\ \sigma_{r\theta} = \sum_p (p+1) r^p \left[pA_p \sin p\theta + (p+2) B_p \sin(p+2)\theta - pC_p \cos p\theta - (p+2) D_p \cos(p+2)\theta \right] \end{cases}$$

Applying the boundary conditions $\sigma_{\theta\theta} = \sigma_{r\theta} = 0$ at $\theta = \pm\pi$

$$\begin{cases} A_p \cos p\pi + B_p \cos(p+2)\pi \pm C_p \sin p\pi \pm D_p \sin(p+2)\pi = 0 \\ \pm pA_p \sin p\pi \pm (p+2) B_p \sin(p+2)\pi - pC_p \cos p\pi - (p+2) D_p \cos(p+2)\pi = 0 \end{cases}$$

These equations can be satisfied 2 ways

- $\sin p\pi = 0 \Rightarrow p =$ integers, $\cos p\pi = \pm 1$

  $$A_p + B_p = 0 \quad \& \quad pC_p + (p+2) D_p = 0$$

- $\cos p\pi = 0 \Rightarrow p = \frac{\text{odd integers}}{2}$, $\sin p\pi = \pm 1$

  $$C_p + D_p = 0 \quad \& \quad pA_p + (p+2) B_p = 0$$

Let's consider the most singular term, we can write

$$\sigma_{\theta\theta} = r^p \tilde{\sigma}_{\theta\theta/p}(\theta), \quad \sigma_{rr} = r^p \tilde{\sigma}_{rr/p}(\theta), \quad \sigma_{r\theta} = r^p \tilde{\sigma}_{r\theta/p}(\theta)$$

$\sigma_{ij} = r^p \tilde{\sigma}_{ij/p}(\theta)$ for Cartesian components. & $\epsilon_{ij} = S_{ijkl} r^p \tilde{\sigma}_{kl/p}(\theta)$

The strain energy density is

$$U=\frac{1}{2}\sigma_{ij}\varepsilon_{ij}=\frac{1}{2}r^{2p}\underbrace{S_{ijkl}\,\tilde{\sigma}_{ij/p}(\theta)\,\tilde{\sigma}_{kl/p}(\theta)}_{2\tilde{U}_p(\theta)}=r^{2p}\,\tilde{U}_p(\theta)$$

The strain energy in some finite region near the crack tip is

$$\int_V U\,dV=\underbrace{\int_{-\pi}^{\pi}\tilde{U}_p(\theta)\,d\theta}_{\text{finite}}\ \underbrace{\int_0^R r^{2p+1}\,dr}\ \sim\begin{cases}\ln r\,\Big|_0^R & \text{for } p=-1 \quad \times\\ r^{2p+2}\,\Big|_0^R & \text{for } p\neq-1\end{cases}$$

$$\rightarrow\ p=\underbrace{-\tfrac{1}{2}}_{\text{gives the most dominant term as } r\to 0},\ 0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ 2\ \ldots\ldots$$

When $p=-\frac{1}{2}$, $C_{-\frac{1}{2}}=-D_{-\frac{1}{2}}$, $B_{-1/2}=\frac{1}{3}A_{-1/2}$ (Two unknowns tbd according to far field)

$$\rightarrow\ \sigma_{\theta\theta}=Ar^{-1/2}\left(\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{3\theta}{2}\right)+Cr^{-1/2}\left(-\frac{3}{4}\sin\frac{\theta}{2}-\frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{r\theta}=Ar^{-1/2}\left(\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{3\theta}{2}\right)+Cr^{-1/2}\left(\frac{1}{4}\cos\frac{\theta}{2}+\frac{3}{4}\cos\frac{3\theta}{2}\right)$$

$$\sigma_{rr}=Ar^{-1/2}\left(\frac{5}{4}\cos\frac{\theta}{2}-\frac{1}{4}\cos\frac{3\theta}{2}\right)+Cr^{-1/2}\left(-\frac{5}{4}\sin\frac{\theta}{2}+\frac{3}{4}\sin\frac{3\theta}{2}\right)$$

A convention due to Irwin calls for

[Actually, Irwin used $\frac{K}{\sqrt{2r}}$, the π showed up later]

$$\sigma_{yy}=\frac{K_I}{\sqrt{2\pi r}}\quad\text{and}\quad\sigma_{xy}=\frac{K_{II}}{\sqrt{2\pi r}}$$

on the plane ahead of the crack (i.e., on $\theta=0$). $\sigma_{\theta\theta}\big|_{\theta=0}=\sigma_{yy}\big|_{x=0}=\frac{A}{r^{1/2}}\rightarrow A=\frac{K_I}{\sqrt{2\pi}}$

$\sigma_{xy}\big|_{x=0}=\sigma_{r\theta}\big|_{\theta=0}=\frac{C}{r^{1/2}}\rightarrow C=\frac{K_{II}}{\sqrt{2\pi}}$

$$\rightarrow \sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \underbrace{\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2} \right)}_{\tilde{\sigma}^{I}_{\theta\theta}} + \frac{K_{II}}{\sqrt{2\pi r}} \underbrace{\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2} \right)}_{\tilde{\sigma}^{II}_{\theta\theta}}$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \underbrace{\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2} \right)}_{\tilde{\sigma}^{I}_{r\theta}} + \frac{K_{II}}{\sqrt{2\pi r}} \underbrace{\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2} \right)}_{\tilde{\sigma}^{II}_{r\theta}}$$

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}} \underbrace{\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2} \right)}_{\tilde{\sigma}^{I}_{rr}} + \frac{K_{II}}{\sqrt{2\pi r}} \underbrace{\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2} \right)}_{\tilde{\sigma}^{II}_{rr}}$$

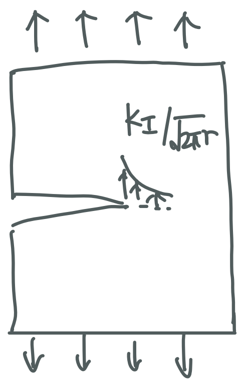
The Cartesian components can be written as

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \left[\cos\frac{\theta}{2} \left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2} \right) \right] + \frac{K_{II}}{\sqrt{2\pi r}} \left[-\sin\frac{\theta}{2} \left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2} \right) \right]$$

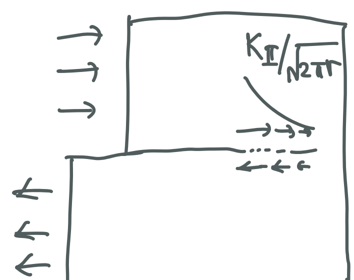
$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \left[\cos\frac{\theta}{2} \left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2} \right) \right] + \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}} \left[\cos\frac{\theta}{2} \left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2} \right) \right]$$

K_I and K_{II} have dimensions of $\sigma L^{1/2}$ and are called the mode I and mode II stress intensity factors. In general, these constants need to be determined based on the speeific loading and geometry of the specimen.

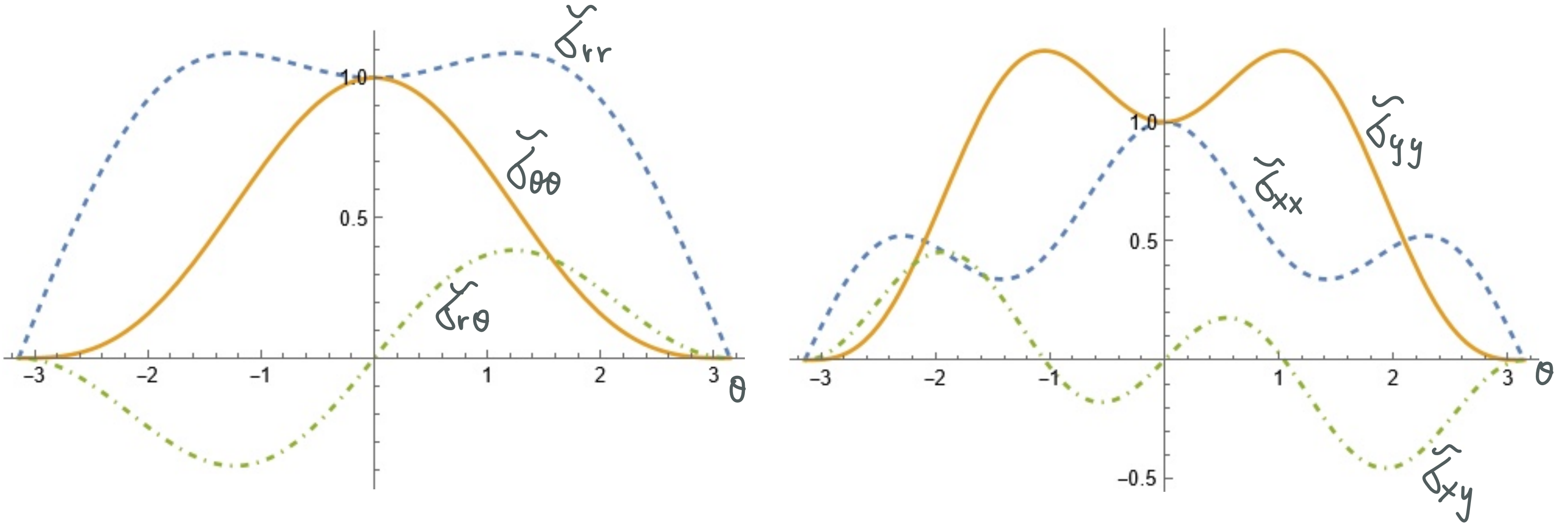


Mode I tensile opening

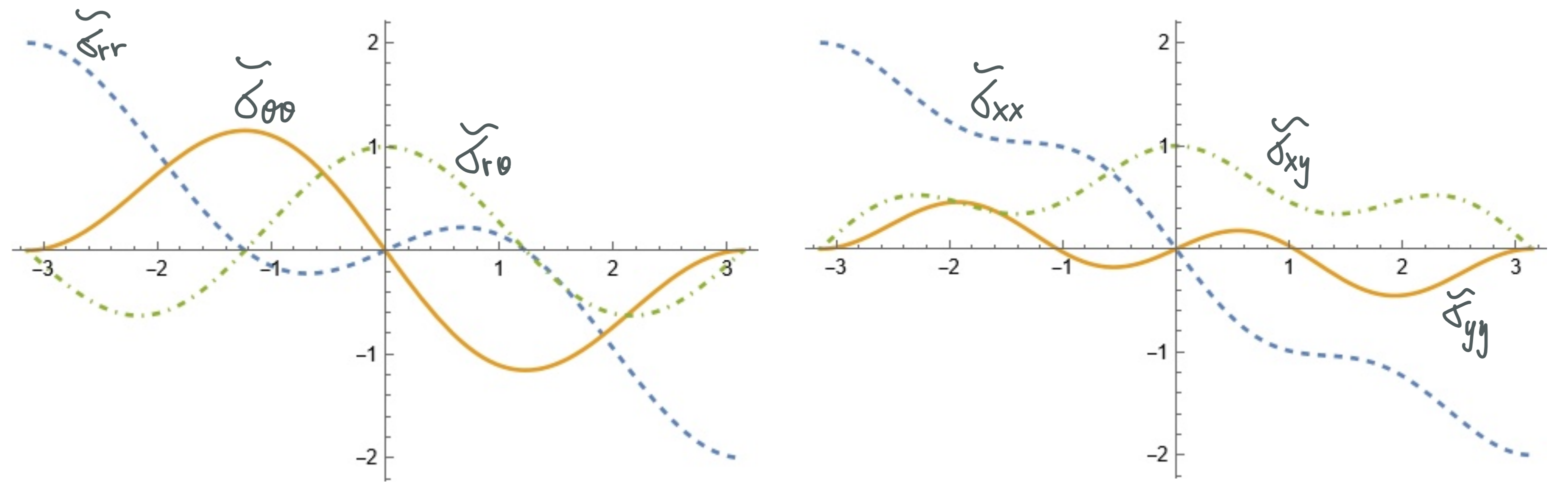


Mode II in-plane shearing

Mode I stress field angular dependence



Mode II stress field angular dependence



The corresponding displacement fields are

$$
\begin{cases}
u_r = \dfrac{K_I}{2E}\sqrt{\dfrac{r}{2\pi}}\,(1+\nu)\left[(2\kappa-1)\cos\dfrac{\theta}{2} - \cos\dfrac{3\theta}{2}\right] + \dfrac{K_{II}}{2E}\sqrt{\dfrac{r}{2\pi}}\,(1+\nu)\left[-(2\kappa-1)\sin\dfrac{\theta}{2} + 3\sin\dfrac{3\theta}{2}\right] \\[2ex]
u_\theta = \dfrac{K_I}{2E}\sqrt{\dfrac{r}{2\pi}}\,(1+\nu)\left[-(2\kappa+1)\sin\dfrac{\theta}{2} + \sin\dfrac{3\theta}{2}\right] + \dfrac{K_{II}}{2E}\sqrt{\dfrac{r}{2\pi}}\,(1+\nu)\left[-(2\kappa+1)\cos\dfrac{\theta}{2} + 3\cos\dfrac{3\theta}{2}\right]
\end{cases}
$$

$$
\begin{cases}
u_x = \dfrac{K_I}{2E}\sqrt{\dfrac{r}{2\pi}}\,(1+\nu)\left[(2\kappa-1)\cos\dfrac{\theta}{2} - \cos\dfrac{3\theta}{2}\right] + \dfrac{K_{II}}{2E}\sqrt{\dfrac{r}{2\pi}}\,(1+\nu)\left[(2\kappa+3)\sin\dfrac{\theta}{2} + \sin\dfrac{3\theta}{2}\right] \\[2ex]
u_y = \dfrac{K_I}{2E}\sqrt{\dfrac{r}{2\pi}}\,(1+\nu)\left[(2\kappa+1)\sin\dfrac{\theta}{2} - \sin\dfrac{3\theta}{2}\right] + \dfrac{K_{II}}{2E}\sqrt{\dfrac{r}{2\pi}}\,(1+\nu)\left[-(2\kappa-3)\cos\dfrac{\theta}{2} - \cos\dfrac{3\theta}{2}\right]
\end{cases}
$$

where $K = \begin{cases} 3-4\nu & \text{plane strain} \\ \frac{3-\nu}{1+\nu} & \text{plane stress.} \end{cases}$

Now, consider the $p=0$ term.

$$\phi = \left(A_0 + B_0 \cos 2\theta + D_0 \sin 2\theta\right) r^2$$

Boundary conditions give $A_0 = -B_0$, $D_0 = 0$

$$\phi = A_0 \left(1 - \cos 2\theta\right) r^2$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = 2A_0(1-\cos 2\theta)$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 2A_0(1-\cos 2\theta) + 4A_0 \cos 2\theta = 2A_0(1+\cos 2\theta)$$

$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} = 2A_0 \sin 2\theta - 4A_0 \sin 2\theta = -2A_0 \sin 2\theta$$

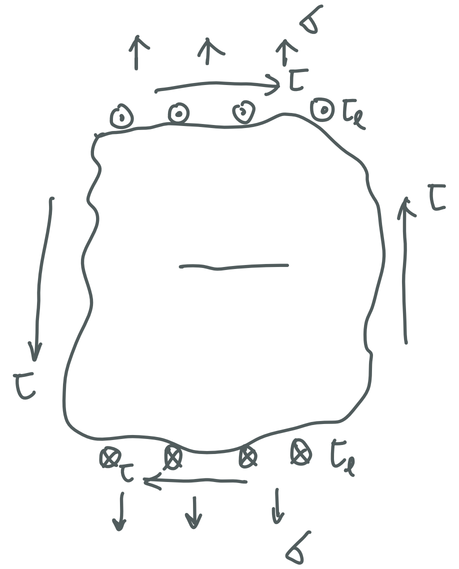
Let $2A_0 = T_{xx}$ by convention, which is called the "T stress". Transforming to Cartisian coordinates gives T stress terms $(O(1))$:

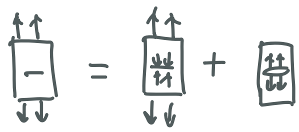
$$\sigma_{xx} = T_{xx}, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = 0$$

Also note that a uniaxial stress in the x_3 direction can be applied and BCs will still be satisfied, so there is a T_{33} (T_{zz}) T-stress term on the order of r^0 as well:

$$\sigma_{zz} = T_{zz}$$

Finally, there exists another mode of crack loading called mode III. This mode is a "tearing" mode and results from anti-plane/longitudinal shear.



We use  before for a center crack

Here we may think of 

To solve the fields very close to the Mode III crack tip, consider the following equations for longitudinal shear in isotropic elasticity

function of x_1, x_2

$$\text{Equilibrium:} \quad \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0$$

$$\text{Kinematics:} \quad \epsilon_{13} = \frac{1}{2}\frac{\partial u_3}{\partial x_1}, \quad \epsilon_{23} = \frac{1}{2}\frac{\partial u_3}{\partial x_2}$$

$$\text{Hooke's law:} \quad \sigma_{13} = 2\mu\,\epsilon_{13}, \quad \sigma_{23} = 2\mu\,\epsilon_{23} \qquad \mu = \frac{E}{2(1+\nu)}$$

In HW2, you will be in charge of finding the asymptotic K_{III} field and any T-stresses.

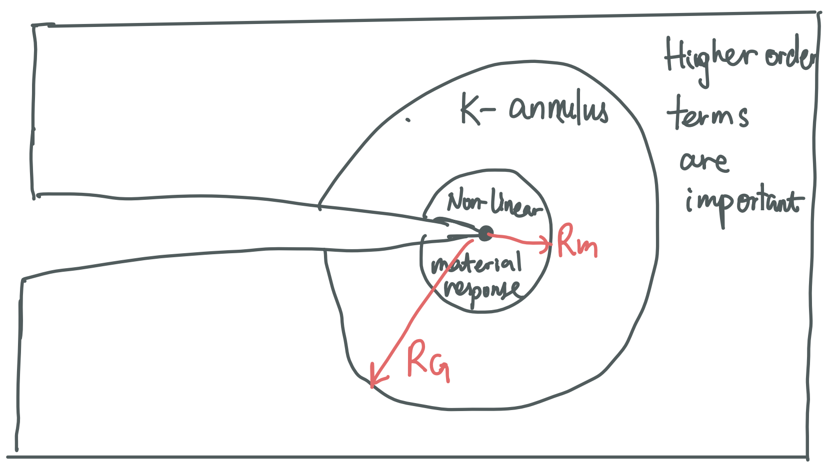
Therefore, stress field near a crack tip can be expanded in the following way

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} \tilde{\sigma}_{ij}^{I}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} \tilde{\sigma}_{ij}^{II}(\theta) + \frac{K_{III}}{\sqrt{2\pi r}} \tilde{\sigma}_{ij}^{III}(\theta) \leftarrow \text{leading order}$$

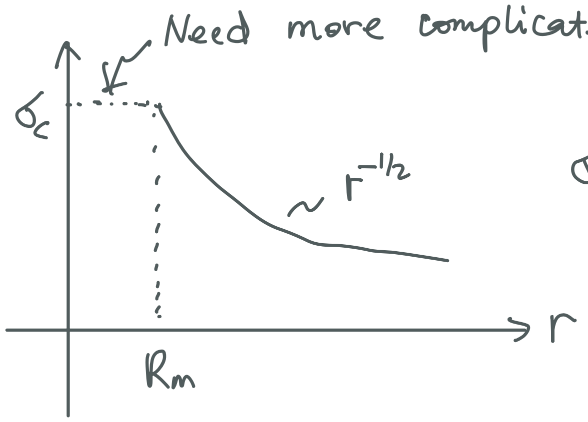
$$+ T_{11}\delta_{i1}\delta_{j1} + T_{33}\delta_{i3}\delta_{j3} + T_{13}(\delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1})$$

$$+ O(r^{1/2}) + O(r) + O(r^{3/2}) + \cdots$$

This gives rise to the idea of "K-annulus", in which the leading order K terms are valid.



• Within the region inside Rm, the assumptions of linear elasticity break down, i.e., physically stresses do not $\to \infty$. This is usually manifested in some type of non-linear material behaviors such as yielding for ductile materials and "peak" stresses observed in the first lecture for "perfectly brittle" materials



$\sigma \sim \frac{K}{\sqrt{2\pi r}} \sim \sigma_c$ (peak stress, yield strength etc)

$\rightarrow \boxed{R_m \sim \frac{1}{2\pi}\left(\frac{K}{\sigma_c}\right)^2}$

- In the region outside R_G, higher-order terms (T stresses and above) arising due to the introduction of a length scale from the specimen geometry become important. (We are able to tell what is meant by $r=1$).

$r^{-1/2} \gg 1$ requires $r \ll \ell$ ← Some length $\rightarrow$ $R_G \sim \frac{1}{10}\min(a, L, \cdots)$

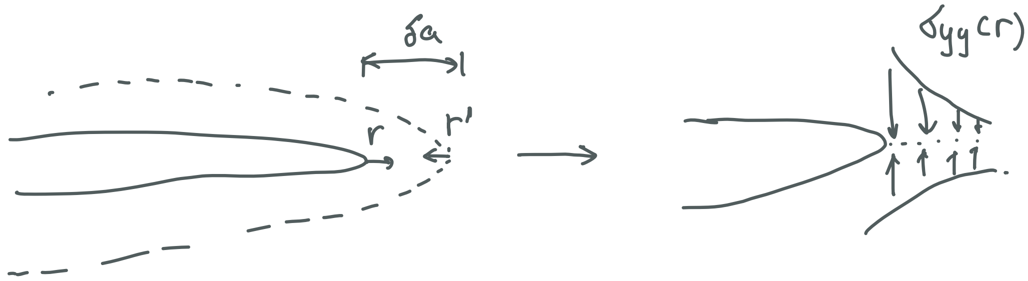
a: crack length; L: specimen dimension etc.

The K – G relationship

Energy release rate is all we asked for from the BVP. Now we have known K_I, K_{II}, K_{III} as "integration constants" to be determined. Before getting to this part, let's determine the relationship between $G \sim \frac{\text{Energy}}{L^2}$ and $K \sim \text{stress} \times L^{1/2}$, while we expect $G \sim K^2/E$

Irwin performed the following "crack closure" integral to determine how much energy is "needed" to "close" the crack tip by an increment of δa for Mode I.

"released" "open"



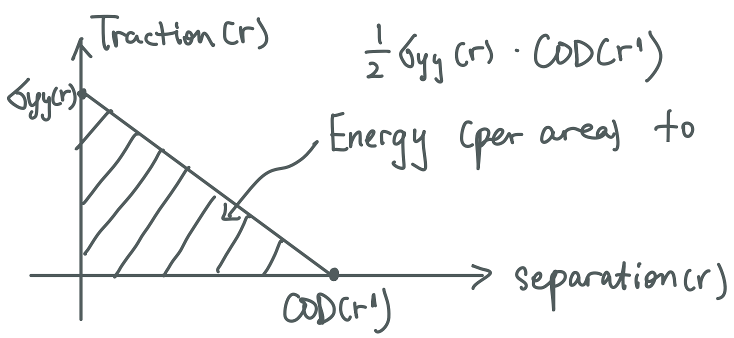
$\sigma_{xy}(r) = 0$

$\sigma_{yy}(r) = \dfrac{K_I}{\sqrt{2\pi r}}$

As we apply such traction, the crack opening displacement (COD) goes from

$$COD = u_y(r', \pi) - u_y(r', -\pi) = \frac{K_I}{E}\sqrt{\frac{r'}{2\pi}}\,(1+\nu)(2\kappa+2)$$

to zero. For any point along the closing region, we should have a linear traction-separation relation



$\frac{1}{2}\sigma_{yy}(r)\cdot COD(r')$

Energy (per area) to close the point at r, or at $r' = \delta a - r$

$$\rightarrow \delta W = \int_0^{\delta a} \frac{1}{2}\sigma_{yy}(r)\cdot COD(\delta a - r)\,dr \times t \quad \leftarrow \text{thickness}$$

$$= \frac{1}{2}\frac{K_I}{\sqrt{2\pi}}\cdot\frac{K_I}{E}\cdot\frac{(1+\nu)(2\kappa+2)}{\sqrt{2\pi}}\int_0^{\delta a}\sqrt{\frac{\delta a - r}{r}}\,dr$$

$r = \delta a \sin^2\theta$

$dr = 2\,\delta a \sin\theta\cos\theta\, d\theta$

$$= \frac{K_I^2 t}{E}\,\frac{(1+\nu)(\kappa+1)}{2\pi}\underbrace{\int_0^{\frac{\pi}{2}} 2\delta a\cdot\frac{\cos\theta}{\cancel{\sin\theta}}\,\cancel{\sin\theta}\cos\theta\,d\theta}_{2\delta a\int_0^{\frac{\pi}{2}}\frac{1}{2}(1+\cos 2\theta)\,d\theta \;=\; \frac{\pi}{2}\delta a}$$

$$= \frac{K_I^2 t \delta a}{E} \frac{(1+\nu)(\kappa+1)}{4} = \mathcal{G}\, \delta a \times t$$

$$\rightarrow \boxed{\mathcal{G} = \frac{K_I^2}{E'} = \begin{cases} \dfrac{K_I^2}{E} & \text{plane stress} \\ \dfrac{K_I^2(1-\nu^2)}{E} & \text{plane strain} \end{cases}}$$

We will find similar results for Mode II and Mode III and obtain

$$\mathcal{G} = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

[Will proof this in HW4 with J- integral method]

for isotropic linear elastic solids (modes are decoupled). Note that stresses and displacements are linear in K but non linear in $\mathcal{G}$. Therefore K values can be added for two superposed elasticity problems / solutions but $\mathcal{G}$ cannot be added in general.

$$K_I = K_I^{(1)} + K_I^{(2)}, \quad \mathcal{G}^{(1)} = \frac{(K_I^{(1)})^2}{E'}, \quad \mathcal{G}^{(2)} = \frac{(K_I^{(2)})^2}{E'}, \quad \mathcal{G} = \frac{K_I^2}{E'} \neq \mathcal{G}^{(1)} + \mathcal{G}^{(2)}$$

One exception is the decoupling of Mode I, II, & III in isotropic elasticity. For anisotropic elasticity

$$\mathcal{G} = \sum_{i=I}^{III} \sum_{j=I}^{IV} K_i H_{ij} K_j$$

where $\underset{\sim}{H} = \begin{bmatrix} 1/E' & 0 & 0 \\ 0 & 1/E' & 0 \\ 0 & 0 & 1/2\mu \end{bmatrix}$ for isotropic elasticity.