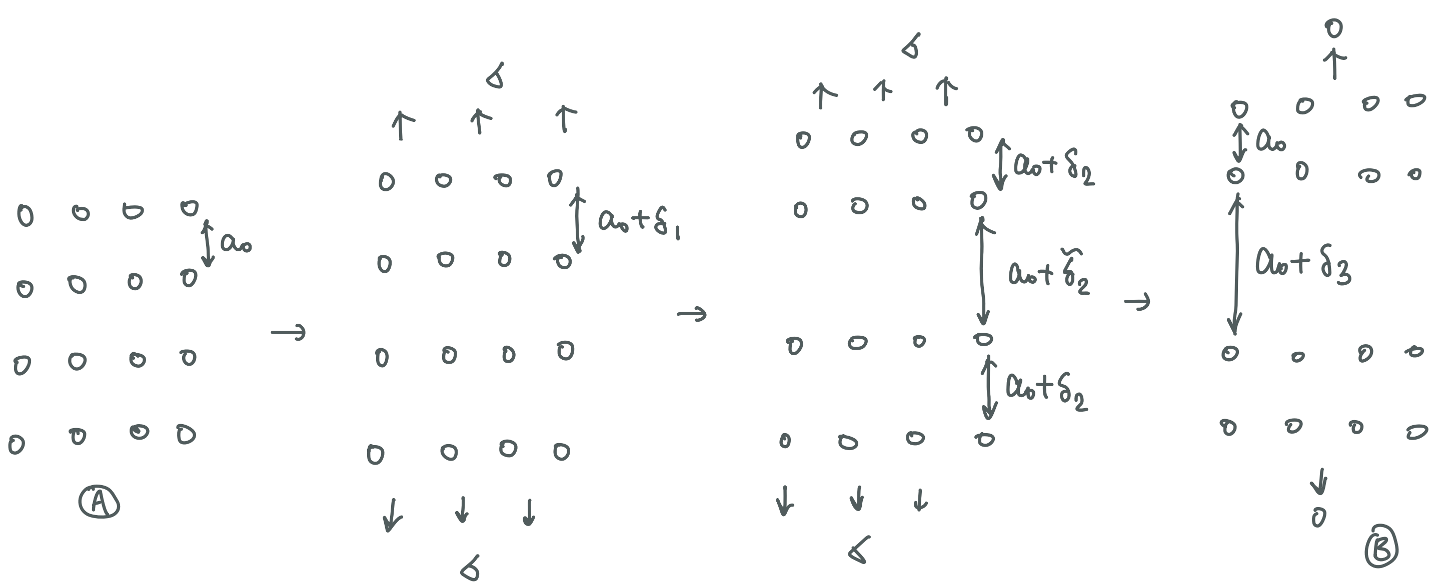
Theoretical tensile strength

The determination of when a body will fail by the growth of a dominant crack requires two things:

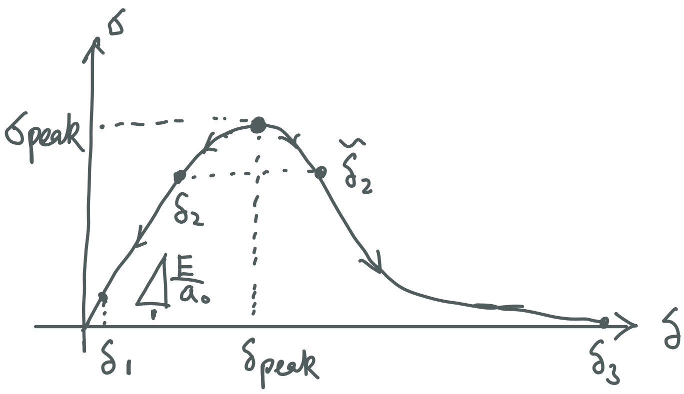
- The determination of the stress and strain fields in the body or, at least, some mechanical quantity (eg., energy) that can characterize the intensity of loading. (This requires solving a boundary value problem)

- A criterion for the crack to advance/propogate. (This distinguishes this course from what you have learnt from linear/nonlinear elasticity).

We will discuss the bvp and the criterion extensively in the class. However, let's first look at an ideal, simplified problem - the strength measured by stretching an infinite atomic lattice to two separated semi-infinite lattices:



The $\sigma - \delta$ curve looks like:



Note $\frac{E}{a_0} \times \delta_{peak} \sim \sigma_{peak}$

$\delta_{peak} \times \sigma_{peak} \sim 2\gamma$

$\rightarrow \delta_{peak} \sim \left(\frac{\gamma a_0}{E}\right)^{1/2}$

$\sigma_{peak} \sim (E\gamma/a_0)^{1/2}$

The stress state are identical between Ⓐ and Ⓑ. But there is a difference in energy

$$2\gamma = \int_0^\infty \sigma \, d\delta$$

↑ surface enery required to create $\underline{\underline{2}}$ new surfaces

Let's estimate σ_{peak} using a "back of the envelop" type of approximation and some material properties that we are familiar with.

Assuming the following form of $\sigma - \delta$ relation:

$$\sigma = A\delta e^{-\delta/\beta}$$

which appears $\sigma \sim \delta$ as $\delta \ll \beta$ and $\sigma \sim e^{-\delta/\beta}$ as $\delta \gg \beta$. To determine A, β, we can use

$$\left.\frac{d\sigma}{d\delta}\right|_{\delta=0} = \frac{E}{a_0} \rightarrow A = \frac{E}{\delta_0}$$

$$\int_0^\infty A\delta e^{-\delta/\beta} = 2\gamma \rightarrow A\left(-\beta\delta e^{-\delta/\beta} - \beta^2 e^{-\delta/\beta}\right)\Big|_0^\infty = 2\gamma \rightarrow \beta = \left(\frac{2\gamma a_0}{E}\right)^{1/2}$$

$$\rightarrow \left.\frac{d\sigma}{d\delta}\right|_{\delta_{peak}} = 0 \rightarrow \delta_{peak} = \beta$$

$$\sigma_{peak} = \sigma(\delta = \delta_{peak}) = \frac{\sqrt{2}}{e}\left(\frac{\gamma E}{a_0}\right)^{1/2} \simeq \frac{1}{2}\left(\frac{\gamma E}{a_0}\right)^{1/2}$$

Some typical values:

$$E \sim 100\ GPa \quad , \quad a_0 \sim 1\ \mathring{A} \quad , \quad \gamma \sim 1\ J/m^2$$

$$\rightarrow \sigma_{peak} \sim \frac{1}{2}\left(\frac{1\times 10^{11}}{10^{-10}}\right)^{1/2} Pa \simeq 15\ GPa$$

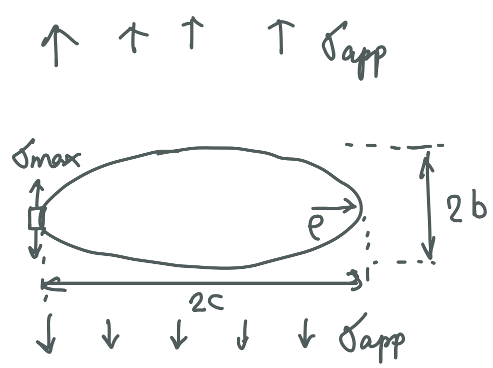
This is a very high strength. Only for nearly perfect crystals (i.e., graphen) $\sigma_{peak} \sim E/10$ is the correct order of magnitude. More generally, it is $\left(\frac{1}{1000} - \frac{1}{100}\right)E$. The question now is why materials are so "weak" or why $E/10$ has been an overestimation !? – Flaws

Strength of materials accounting for flaws

Recall from elasticity theory that the stress concentration near an elliptical flaw is given as:

$$\frac{\sigma_{max}}{\sigma_{app}} = 1 + 2\frac{c}{b}$$

–Inglis (1913)



The radius of curvature of the ellipse is given as $\rho = \frac{b^2}{c}$. We then have

$\frac{1}{\rho} \sim K(s) = \frac{|\alpha'(s) \times \alpha''(s)|}{|\alpha'(s)|^3}$

$$\sigma_{max} = \sigma_{app}\left(1 + 2\sqrt{\frac{c}{\rho}}\right) \simeq 2\sigma_{app}\sqrt{\frac{c}{\rho}} \quad \text{as} \quad c \gg \rho \text{ or } c \gg b$$

Now change the failure criterion from $\sigma_{app} = \sigma_{peak}$ to $\sigma_{max} = \sigma_{peak}$, i.e.,

$$2\sigma_{app}\left(\frac{c}{\rho}\right)^{1/2} = \frac{1}{2}\left(\frac{E\gamma}{a_0}\right)^{1/2} \rightarrow \sigma_{failure} = \frac{1}{4}\left(\frac{E\gamma}{c}\cdot\frac{\rho}{a_0}\right)^{1/2} \simeq \underbrace{\frac{1}{4}\left(\frac{E\gamma}{c}\right)^{1/2}}$$

for the sharpest crack/flaw that is likely to propagate first.

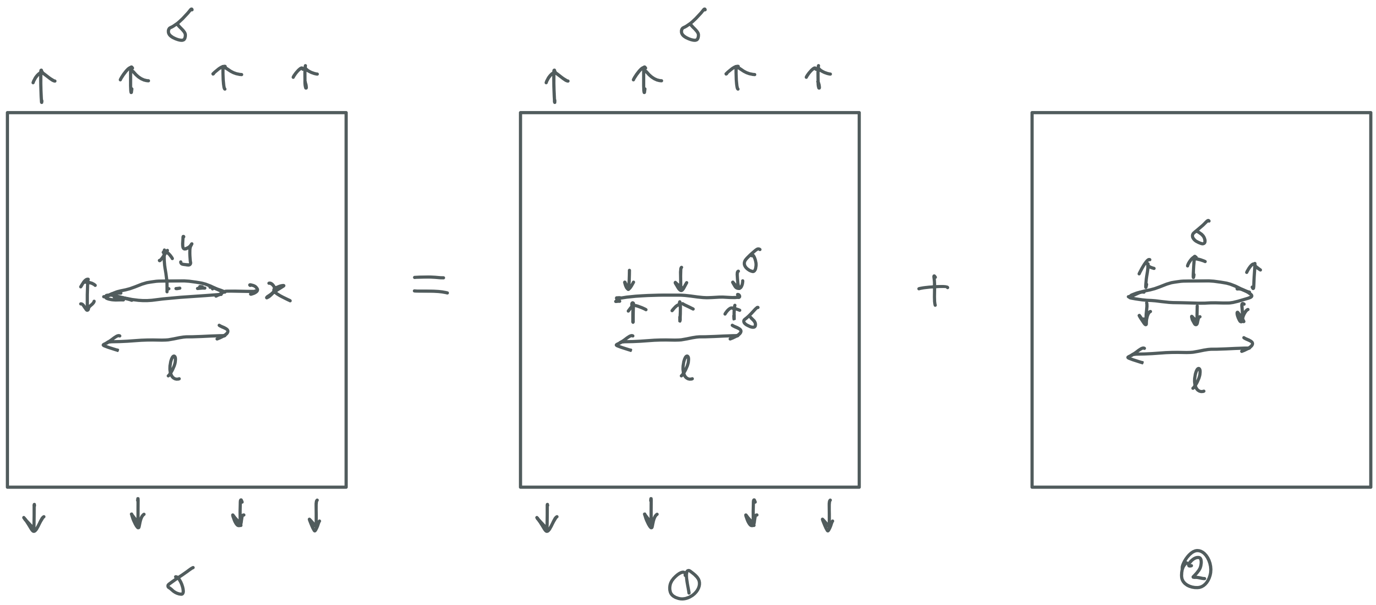
- If we take $c \sim 1\,\mu m$, we can have $\sigma_{failure} \sim 75$ MPa $\sim \frac{E}{1000}$. This is close to the range for materials we are familiar with.

- We have assumed the material does not deform plastically. Hence this model is most applicable to rocks, glasses and ceramics at relatively low temperature.

- Note the $1/\sqrt{c}$ dependence of the strength. For brittle materials, failure strength is <u>not</u> a material property. "How to measure? – Introduce notches that are large enough."

Griffith theory – An energy approach

Assumptions: linear elastic, isotropic, homogeneous, perfectly brittle, small cracks, plane strain



We want to find the potential energy of the system with respect to crack length l. Consider the problem as the superposition of problem ① and problem ②

The total potential energy of the system is given as

$$W = \underbrace{W^{①} + W^{②}}_{\text{Incorrect in general}} = W^{①} + \underbrace{W_{SE} - W_{\sigma}}_{\text{linearity}} = W^{①} - \tfrac{1}{2} W_{\sigma}$$

The total energy of the system (per thickness) is

$$\phi = W + 2\gamma l = \cancelto{0}{W^{①}} - \tfrac{1}{2} W_{\sigma} + 2\gamma l$$

since it is energy of a uniform stress problem independent of l

or you may think it is a reference at $l=0$!

To determine W_σ, recall from elasticity theory that the crack opening displ. is given as

$$\delta = \frac{4\sigma(1-\nu^2)}{E}\sqrt{\frac{l^2}{4}-x^2}$$

We will also derive this by solving the BVP shortly. The work done by σ in the auxiliary problem 2 is then simply

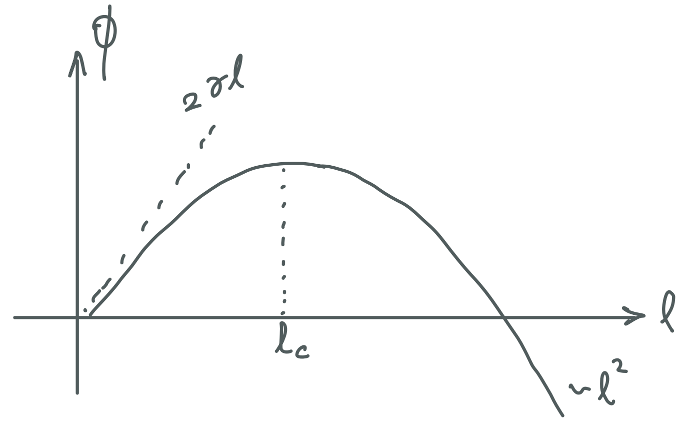
$$W_\sigma = \int_{-l/2}^{l/2}\sigma\delta\,dx = \frac{4\sigma^2(1-\nu^2)}{E}\int_{-l/2}^{l/2}\sqrt{\frac{l^2}{4}-x^2}\,dx = \frac{\pi}{2}\frac{1-\nu^2}{E}\sigma^2 l^2$$

[Chatgpt gives correct steps but a wrong answer by a factor of 2 ...

$$\int_{-l/2}^{l/2}\sqrt{\frac{l^2}{4}-x^2}\,dx \;\underset{dx=\frac{l}{2}\cos\theta\,d\theta}{\overset{x=\frac{l}{2}\sin\theta}{=\!=\!=}}\; \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{l^2}{4}\cos^2\theta\,d\theta = \frac{1}{8}l^2$$

Back to the total energy of the system

$$\boxed{\phi = 2\gamma l - \frac{\pi}{4}\frac{1-\nu^2}{E}\sigma^2 l^2}$$



- The system will be in thermodynamic equilibrium (not stable though) when

$$\left.\frac{\partial\phi}{\partial l}\right|_\sigma = 0 \rightarrow 2\gamma - \frac{\pi}{2}\frac{1-\nu^2}{E}\sigma^2 l_c = 0 \rightarrow l_c = \frac{4}{\pi}\frac{\gamma E}{(1-\nu^2)\sigma^2}$$

- Under prescribed σ, cracks with $l > l_c$ (i.e., $\frac{\partial \phi}{\partial l} < 0$) will grow while cracks with $l < l_c$ (i.e., $\frac{\partial \phi}{\partial l} > 0$) will "heal". In air, this does not actually happen because once a crack forms, a barrier such as oxide and passivation layer is created spontaneously. In vacuum, crack healing can and does occur. For instance, in space, designers have to deal with the problem of cold welding

- For a given l, the critical stress for the crack to propagate is

$$\sigma_c = \left(\frac{4}{\pi} \frac{E}{1-\nu^2} \cdot \frac{\gamma}{l} \right)^{1/2},$$

which again leads to a $1/\sqrt{l}$ dependence of strength on crack length (Note that this is based on an energy approach).

Define the "driving force" for crack propagation — energy release rate $\mathcal{G}$

$$\boxed{\mathcal{G} = -\frac{\partial W}{\partial A} = -\frac{\partial W}{\partial (l \times t)}}$$

Potential energy

thickness

such that crack propagation occurs when $\mathcal{G} = 2\gamma$ for "perfectly brittle" solids.

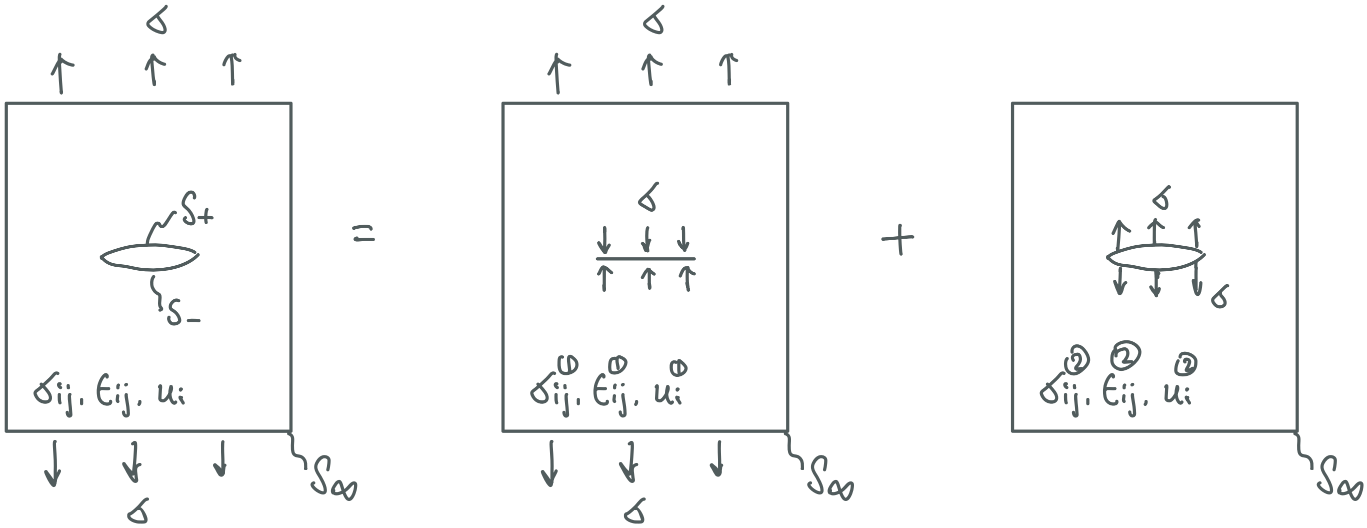
"Resisting force"

The crack growth is unstable when $\partial \mathcal{G}/\partial l > 0$ [$\partial \mathcal{G}/\partial l = \pi \sigma^2 (1-\nu^2)/E$ for this problem. Alternatively, the crack growth is stable when $\partial \mathcal{G}/\partial l < 0$ and neutrally stable when $\partial \mathcal{G}/\partial l = 0$.

Let's return to our calculation of the strain energy in our problem.

Under given identical BCs, we have $\sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}$ and $W \neq W^{(1)} + W^{(2)}$.

Why did it work for our problem?



$$W_{SE} = \int_V \frac{1}{2}\sigma_{ij}\epsilon_{ij}\,dV = \int_S \frac{1}{2} T_i u_i \,ds \quad \text{(by principle of virtual work)}$$

$$= \int_S \frac{1}{2}(T_i^{(1)} + T_i^{(2)})(u_i^{(1)} + u_i^{(2)})\,ds$$

$$= W_{SE}^{(1)} + W_{SE}^{(2)} + \underbrace{\int_S \frac{1}{2} T_i^{(1)} u_i^{(2)}\,ds + \int_S \frac{1}{2} T_i^{(2)} u_i^{(1)}\,ds}_{\text{The two are identical by reciprocal theorem}}$$

Let's evaluate $\int_S \frac{1}{2} T_i^{(2)} u_i^{(1)}\,ds$ since $\underset{\sim}{T}^{(2)}$ is simply $\underset{\sim}{0}$ at $S = S_\infty$.

$$\int_S T_i^{(2)} u_i^{(1)}\,ds = \int_{S+} T_i^{(2)} u_i^{(1)}\,ds + \int_{S-} T_i^{(2)} u_i^{(1)}\,ds = 0,$$

since $T_i^{(2)}$ on $S_+ = -T_i^{(2)}$ on S_- while $u_i^{(1)}$ on $S_+ = u_i^{(1)}$ on S_-.

→ $W_{SE} = W_{SE}^{(1)} + W_{SE}^{(2)}$ for our problem. This is not true in general. However, we have had this for $W_{SE} = W_{Bending} + W_{Stretching} + W_{Torsion}$.

Energy release rate

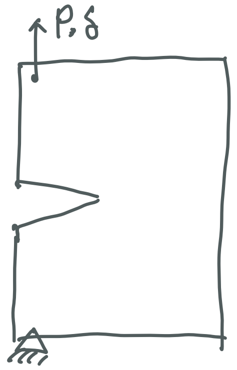
Recall: $\mathcal{G} = -\dfrac{\partial W}{\partial l}$ (in 2D)

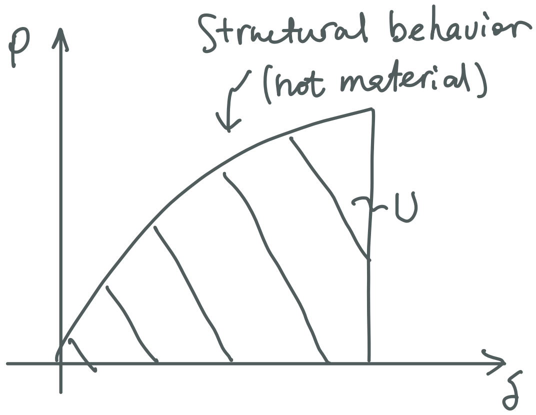
W = stored strain energy − work done by loads (per unit thickness)

l = total crack length

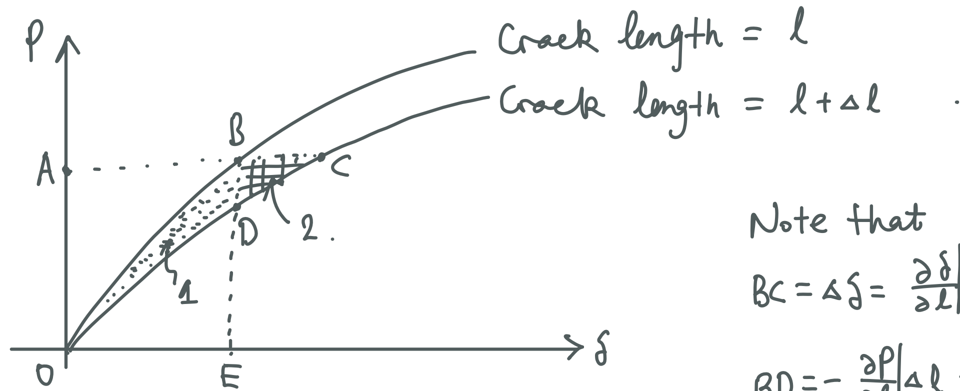
Consider a non-linear elastic body with a crack subject to point loading.

per unit thickness





Since $U = \int_0^\delta P\, d\delta$, $P = \partial U / \partial \delta$. Besides, $U(\delta, l)$ is a function of both δ & l.
Now consider a specimen with a crack length of $l + \Delta l$. The structural behavior will be "softer".



Note that

$BC = \Delta\delta = \left.\dfrac{\partial \delta}{\partial l}\right|_P \Delta l$

$BD = -\left.\dfrac{\partial P}{\partial l}\right|_\delta \Delta l = -\dfrac{\partial^2 U}{\partial \delta \partial l} \Delta l.$

$\rightarrow BCD \simeq -\frac{1}{2} \dfrac{\partial^2 U}{\partial \delta \partial l} \Delta l \cdot \dfrac{\partial \delta}{\partial l} \Delta l \sim O(\Delta l^2)$

• Displacement control

$W(l) = OBE$, $W(l+\Delta l) = ODE$, $\mathcal{G} = -\Delta W/\Delta l = OBD/\Delta l$.

"Released strain energy"

$$W(l) = U(\delta, l)$$

$$W(l+\Delta l) = U(\delta, l+\Delta l) = U(\delta, l) + \left.\frac{\partial U}{\partial l}\right|_{\delta} \Delta l + O(\Delta l^2)$$

$$\Delta W = W(l+\Delta l) - W(l) = \left.\frac{\partial U}{\partial l}\right|_{\delta} \Delta l + O(\Delta l^2)$$

$$\rightarrow \mathcal{G} = \lim_{\Delta l \to 0} -\frac{\Delta W}{\Delta l} = -\left.\frac{\partial U}{\partial l}\right|_{\delta}$$

- Load control

$W(l) = -OAB$, $W(l+\Delta l) = -OAC$, $\mathcal{G} = -\Delta W/\Delta l = OBC/\Delta l$ (will show $BCD \sim \Delta l^2$).

$$W(l) = U(\delta, l) - P\delta$$

$$W(l+\Delta l) = U(\delta+\Delta\delta, l+\Delta l) - P(\delta+\Delta\delta)$$

$$= U(\delta, l) + \cancel{\underbrace{\left.\frac{\partial U}{\partial \delta}\right|_{l}}_{P}\Delta\delta} + \left.\frac{\partial U}{\partial l}\right|_{\delta}\Delta l - P\delta - \cancel{P\Delta\delta} + \underbrace{O(\Delta l^2)}$$

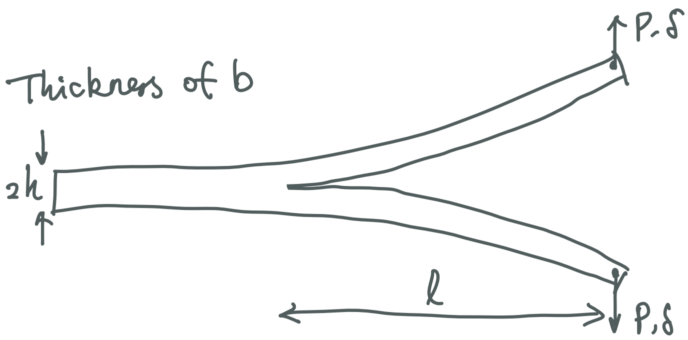
Note $\Delta\delta = \frac{\partial \delta}{\partial l}\Delta l$

$$\Delta W = \left.\frac{\partial U}{\partial l}\right|_{\delta} \Delta l + \Delta(\Delta l^2)$$

$$\rightarrow \mathcal{G} = \lim_{\Delta l \to 0} -\frac{\Delta W}{\Delta l} = -\left.\frac{\partial U}{\partial l}\right|_{\delta}$$ (so $BCD \sim \Delta l^2$, back to the figure)

For point loading, $\mathcal{G}$ is the same under load or displacement control. For more general loading it can also be shown that $\mathcal{G}$ does not depend on the loading conditions. On the other hand, crack growth stability does depend on them!

Ex) The double cantilever beam specimen



$l \gg h$ (the crack tip is not too close to the end of the specimen)

We can treat the specimen as two cantilever beams of length l.

$$U_0 = \frac{1}{2} P\delta = \int_0^l \frac{1}{2}\sigma\epsilon \, dV \qquad \left(\text{Note } \sigma = \frac{My}{I}, \ \epsilon = \frac{\sigma}{E}\right)$$

$$= \int_0^l \frac{1}{2}\frac{M^2}{EI^2} y^2 \, dV$$

$$= \int_0^l \frac{1}{2}\frac{M^2}{EI} \, dx \qquad \left(\text{or } \int_0^l \frac{1}{2} M K \, dx, \ M = Px\right)$$

$$= \frac{1}{6}\frac{P^2 l^3}{EI} = \frac{2P^2 l^3}{Ebh^3}$$

$$\rightarrow P = \frac{1}{4}\frac{Ebh^3}{l^3}\cdot\delta$$

- Fixed P

$$W = 2U - 2P\delta = -\frac{4P^2 l^3}{Ebh^3}$$

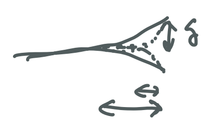
$$\mathcal{G} = -\frac{\partial W}{\partial (bl)} = \frac{12P^2 l^2}{Eb^2h^3}$$

Stability: $\left.\frac{\partial \mathcal{G}}{\partial l}\right|_P = \frac{24P^2 l}{Eb^2h^3} > 0 \rightarrow$ Unstable!

- Fixed δ

$$W = 2U - \cancelto{0}{\text{Work done by } P}$$

{ since δ is fixed
Or we can set the reference energy at this level of δ



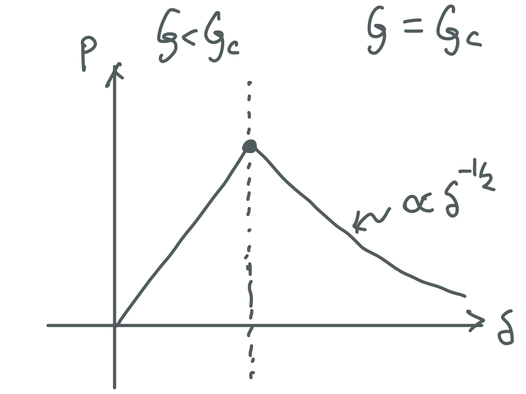
$$= \frac{4P^2 l^3}{Ebh^3} \left(= \frac{4l^3}{Ebh^3}\frac{1}{16}\frac{(Ebh^3)^2}{l^6}\delta^2 \right)$$

$$= \frac{1}{4}\frac{Ebh^3}{l^3}\delta^2$$

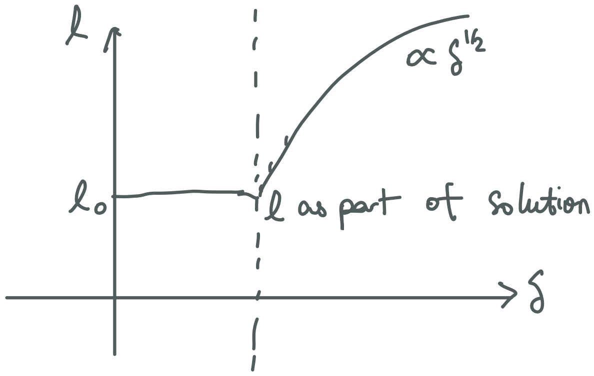
$$\mathcal{G} = -\left.\frac{\partial W}{\partial (lb)}\right|_\delta = \frac{3}{4}\frac{Eh^3}{l^4}\delta^2 = \frac{3}{4}\frac{Eh^3}{l^4}\left[16\frac{l^6}{(Ebh^3)^2}P^2\right] = \frac{12P^2l^2}{Eb^2h^3} \checkmark$$

Stability: $\left.\frac{\partial G}{\partial \ell}\right|_{\delta} = -3\frac{Eh^3\delta^2}{\ell^5} < 0 \rightarrow$ Stable!

Let's move on to experimentally accessible quatitaties, including P, δ, ℓ.



$$\begin{cases} \frac{12P^2\ell^2}{Eb^2h^3} = G_c \\ \frac{3}{4}\frac{Eh^3}{\ell^4}\delta^2 = G_c \end{cases} \Rightarrow \quad P = \left(\frac{Eh^3b^4G_c^3}{108}\right)^{1/4}\cdot\delta^{-\frac{1}{2}}, \quad \ell = \left(\frac{3Eh^3}{4G_c}\right)^{1/4}\delta^{1/2}$$



What is meant by $\ell^2 \propto \delta$, try scaling arguments:

$$K \sim \delta/\ell^2$$

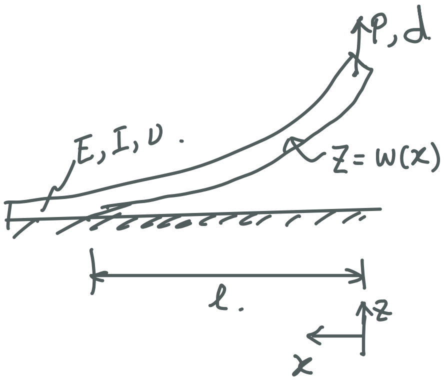
$$P\times\delta \sim G_c \times b\ell \sim EIK^2\times\ell$$

$$\longrightarrow K \sim \left(\frac{G_c b}{EI}\right)^{1/2} \qquad \text{(HW1)}$$

L1 – Introduction

L2 + L3 & HW 1

To show the concept of Griffith's theory further, let's consider the following form of analysis.



Per width

$$\mathcal{F} = \int_0^l \frac{1}{2} EI\kappa^2 dl + \underbrace{\Gamma}_{\gamma_1+\gamma_2} l - \underbrace{P\cancel{d}}_{\text{Fixed } d}$$

$$\simeq \int_0^l \frac{1}{2} EI (w'')^2 dl + \Gamma l$$ ← l is also part of solution

$$\delta \mathcal{F} = \underbrace{\int_0^l EI w'' \delta w'' dl}_{①} + \frac{1}{2} EI w''^2(l)\, \delta l + \Gamma\, \delta l.$$

$$① = \int_0^l EI w'' d\delta w'$$

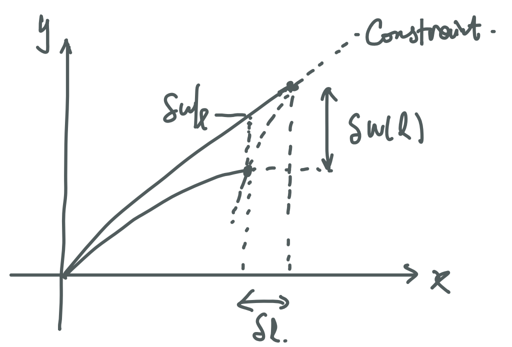
$$= EI w'' \delta w' \Big|_0^l - \int_0^l EI w''' d\delta w$$

$$= EI w'' \delta w' \Big|_0^l - EI w''' \delta w \Big|_0^l + \int_0^l EI w'''' \delta w\, dx$$

$$\Rightarrow \delta \mathcal{F} = \int_0^l EI w'''' \delta w\, dx + \underbrace{EI w'' \delta w' \Big|_l}_{?} - \underbrace{\cancel{EI w'' \delta w' \Big|_0}}_{0 \text{ since } w''(0)=0} - \underbrace{\cancel{EI w''' \delta w \Big|_l}}_{0 \text{ since } w(l)\equiv 0} + \underbrace{\cancel{EI w''' \delta w \Big|_0}}_{0 \text{ since } w(0)\equiv d}$$

$$+ \left(\frac{1}{2} EI w''(l) + \Gamma\right) \delta l.$$

Note $\delta w(l) = \delta w|_l + w' \delta l$. → $\delta w'(l) = \delta w'|_l + w''(l)\, \delta l \equiv 0$ (since $w'(l) \equiv 0$)



$$\Rightarrow \delta \mathcal{F} = \int_0^l EI w'''' \delta w\, dx + \left(\Gamma - \frac{1}{2} EI (w''(l))^2\right) \delta l$$

δw, δl 任意

$\Rightarrow EI w'''' = 0$ subject to (BVP)

$w(0) = d$, $w''(0) = 0$, $w(l) = 0$, $w'(l) = 0$

& $\kappa(l) = w''(l) = (2\Gamma/EI)^{1/2}$ (Criterion)

Mathematica: << VariationalMethods`

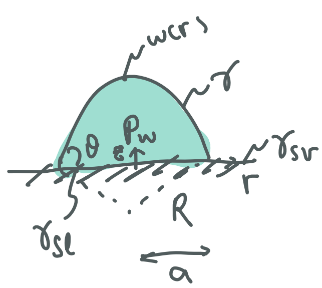
EulerEquations[$\frac{1}{2}$ EI (w''[x])^2 − p · w[x] , w[x], x]

Does not change the local criterion

Questions: ① What if there is external loading p(x)

② What if there are both bending and torsion

This Griffith concept is not limited. You would find a wide range of problems that accounting for surface energies.



$$F = \gamma A - \Delta p V + (\gamma_{sl} - \gamma_{sv})\pi a^2$$

$$F = \cancel{2\pi}\gamma \int_0^a \sqrt{1+w'^2}\, r\,dr - \Delta p \int_0^a \overset{\text{drop.}}{\cancel{2\pi}}\, w\, r\,dr + \overset{\text{drop}}{\cancel{2\pi}}(\gamma_{sl} - \gamma_{sv})\tfrac{1}{2}a^2$$

$$\delta F = \underbrace{\gamma \int_0^a \frac{w'\,\delta(w')}{\sqrt{1+w'^2}}\, r\,dr}_{①} - \Delta p \int_0^a \delta w\, r\,dr + \underbrace{\left(\gamma\sqrt{1+w'^2}\cdot a + \overset{0}{\cancel{\Delta p\, w\, a}}\right)\Big|_{r=a} \delta a}_{②}$$

$$+ (\gamma_{sl} - \gamma_{sv})\, a\,\delta a$$

$$① = \gamma \int_0^a \frac{w'}{\sqrt{1+w'^2}}\, r\, d(\delta w)$$

$$= \underbrace{\frac{\gamma w'}{\sqrt{1+w'^2}}\, r\,\delta w\Big|_0^a}_{③} - \int_0^a r\,\delta w \left[\underbrace{\frac{w''}{\sqrt{1+w'^2}} - \frac{w'^2 w''}{(1+w'^2)^{3/2}}}\right] dr - \int_0^a \frac{\gamma w'}{\sqrt{1+w'^2}}\,\delta w\,dr$$

$$\rightarrow \frac{w''}{(1+w'^2)^{3/2}}$$

$$② = \gamma\sqrt{1+\tan^2\theta}\cdot a\,\delta a = \frac{\gamma a}{\cos\theta}\,\delta a$$

θ ... $\tan\theta = -w'$

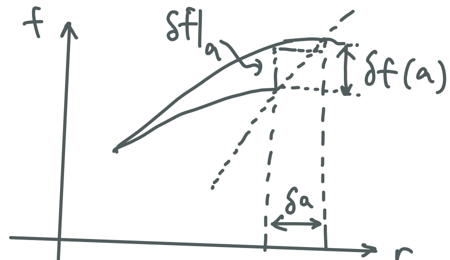
$$③ = \frac{\gamma w'}{\sqrt{1+w'^2}}\, r\,\delta w\Big|_a - \frac{\cancel{\gamma w' r}^{\,0}}{\sqrt{1+w'^2}}\,\delta w\Big|_0$$

$$= -\frac{\gamma w'^2(a)}{\sqrt{1+w'^2(a)}}\, a\,\delta a$$

$$= -\frac{\gamma \tan^2\theta}{\sqrt{1+\tan^2\theta}}\, a\,\delta a = -\frac{\gamma \sin^2\theta}{\cos\theta}\, a\,\delta a$$

$$\delta w(a) = \delta w\Big|_a + w'(a)\,\delta a = 0$$

$$\rightarrow \delta w\Big|_a = -w'(a)\,\delta a$$



$$\delta f(a) = \delta f\Big|_a + (f'(a) + \cancel{\varepsilon}^{\,0})\,\delta a$$

What is this?

$$\delta F = -\int_0^a \left[\frac{\gamma w''}{(1+w'^2)^{3/2}} + \frac{\gamma w'}{r(1+w'^2)^{1/2}} + \Delta p \right] r\,\delta w\, dr$$

$$+ \left(\frac{\gamma}{\cos\theta} - \frac{\gamma \sin^2\theta}{\cos\theta} + \gamma_{sl} - \gamma_{sv} \right) a\,\delta a$$

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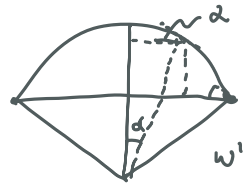

$$w' = -\tan\alpha$$

- $\dfrac{w'}{r(1+w'^2)^{1/2}} = -\dfrac{\sin\alpha}{r} = -\dfrac{1}{R}$

$$w'' = -\frac{d\tan\alpha}{dr} = -\frac{1}{\cos^2\alpha}\frac{d\alpha}{dr}$$

- $\dfrac{w''}{(1+w'^2)^{3/2}} = -\dfrac{d\sin\alpha}{dr} = \dfrac{-1}{R}$

Arbitary $\delta w, \delta a \Rightarrow$ $\boxed{\Delta p = -\gamma\left[\frac{w''}{(1+w'^2)^{3/2}} + \frac{w'}{r(1+w'^2)^{1/2}}\right],\quad \cos\theta = \frac{\gamma_{sv} - \gamma_{sl}}{\gamma}}$

Young-Laplace equation

Young's law (Criterion)

$$\left\{ \begin{array}{l} \cdot \int w\, dA = V \\ \cdot\ w(a) = 0 \\ \cdot\ w'(0) = 0 \end{array} \right.$$

Have been used during the course of variation